UNIFORM DOMAINS AND THE QUASI-HYPERBOLIC METRIC

By

F. W. GEHRING† AND B. G. OSGOOD

Dedicated to the memory of Professor Zeev Nehari

1. Introduction

We shall assume throughout this paper that D and D' are proper subdomains of euclidean n-space R^n, $n \geqq 2$.

We say that D is a *uniform domain* if there exist constants a and b such that each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which

$$(1.1) \qquad \begin{cases} s(\gamma) \leqq a\,|x_1 - x_2|, \\ \min_{j=1,2} s(\gamma(x_j, x)) \leqq b\,d(x, \partial D) \quad \text{for all } x \in \gamma. \end{cases}$$

Here $s(\gamma)$ denotes the euclidean length of γ, $\gamma(x_j, x)$ the part of γ between x_j and x, and $d(x, \partial D)$ the euclidean distance from x to ∂D.

Next for each $x_1, x_2 \in D$ we set

$$k_D(x_1, x_2) = \inf_\gamma \int_\gamma d(x, \partial D)^{-1} ds,$$

where the infimum is taken over all rectifiable arcs γ joining x_1 and x_2 in D. We call k_D the *quasi-hyperbolic metric* in D. From lemma 2.1 in [6] it follows that

$$(1.2) \qquad \begin{cases} \left| \log \dfrac{d(x_1, \partial D)}{d(x_2, \partial D)} \right| \leqq k_D(x_1, x_2), \\ \log \left(\dfrac{|x_1 - x_2|}{d(x_j, \partial D)} + 1 \right) \leqq k_D(x_1, x_2), \quad j = 1, 2, \end{cases}$$

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for all $x_1, x_2 \in D$. Hence

$$j_D(x_1, x_2) \leqq k_D(x_1, x_2), \tag{1.3}$$

where

$$j_D(x_1, x_2) = \frac{1}{2} \log \left(\frac{|x_1 - x_2|}{d(x_1, \partial D)} + 1\right)\left(\frac{|x_1 - x_2|}{d(x_2, \partial D)} + 1\right).$$

Finally when $n = 2$, we say that D is *quasiconformally decomposable* if there exists a constant K with the following property. For each $x_1, x_2 \in D$ there exists a subdomain D_0 of D such that $x_1, x_2 \in \bar{D}_0$ and such that ∂D_0 is a *K-quasiconformal circle*, i.e., the image of the unit circle under a K-quasiconformal mapping of $\bar{R}^2$ onto itself [15]. Here $\bar{R}^n = R^n \cup \{\infty\}$.

Uniform domains were introduced recently in [11] and [12] by O. Martio and J. Sarvas in connection with approximation and injectivity properties of functions defined in domains in R^n. P. W. Jones studied in [8] the domains D for which there exist constants c and d such that

$$k_D(x_1, x_2) \leqq c j_D(x_1, x_2) + d \tag{1.4}$$

for all $x_1, x_2 \in D$; it is precisely this class of domains D for which each function u with bounded mean oscillation in D has an extension v with bounded mean oscillation in R^n.

We show in this paper that a domain D is uniform if and only if it satisfies (1.4) for some constants c and d; hence the two classes of domains mentioned in the above paragraph are identical. This first characterization follows from properties of the quasi-hyperbolic geodesics established in section 2. In section 3 we show that k_D and j_D are quasi-invariant under quasiconformal mappings of D and $\bar{R}^n$, respectively. This fact, together with the above characterization, immediately implies the invariance of the class of uniform domains under quasiconformal mappings of $\bar{R}^n$.

In section 4 we obtain a second characterization for uniform domains in R^2, namely that a domain D in R^2 is uniform if and only if it is quasiconformally decomposable. We then apply this characterization to give an alternative proof for the main injectivity properties of uniform domains in R^2. Finally in section 5 we exhibit a domain D in R^2 which has these injectivity properties but which is not itself uniform.

2. Quasi-hyperbolic metric in uniform domains

We show here that a domain D is uniform if and only if it satisfies inequality (1.4). The necessity is an immediate consequence of the following result.

Theorem 1. *Suppose that $x_1, x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which*

$$(2.1) \qquad \begin{cases} s(\gamma) \leqq a\,|x_1 - x_2|, \\ \min_{j=1,2} s(\gamma(x_j, x)) \leqq b d(x, \partial D) \quad \text{for all } x \in \gamma. \end{cases}$$

Then

$$(2.2) \qquad k_D(x_1, x_2) \leqq c j_D(x_1, x_2) + d$$

where $c = 2b$ *and* $d = 2(b + b \log a + 1)$.

Proof. Choose $x_0 \in \gamma$ so that $s(\gamma(x_1, x_0)) = s(\gamma(x_2, x_0))$. Then by the triangle inequality it is sufficient to show that

$$(2.3) \qquad k_D(x_j, x_0) \leqq b \log\left(\frac{|x_1 - x_2|}{d(x_j, \partial D)} + 1\right) + b(1 + \log a) + 1$$

for $j = 1, 2$. By symmetry we may assume that $j = 1$.

Suppose first that

$$(2.4) \qquad s(\gamma(x_1, x_0)) \leqq \frac{b}{b+1}\, d(x_1, \partial D).$$

If $x \in \gamma(x_1, x_0)$, then

$$d(x, \partial D) \geqq d(x_1, \partial D) - s(\gamma(x_1, x)) \geqq \frac{1}{b+1}\, d(x_1, \partial D)$$

and we obtain

$$k_D(x_1, x_0) \leqq (b+1)\, \frac{s(\gamma(x_1, x_0))}{d(x_1, \partial D)} \leqq b.$$

This implies (2.3) since $a \geqq 1$.

Suppose next that (2.4) does not hold and choose $y_1 \in \gamma(x_1, x_0)$ so that

$$s(\gamma(x_1, y_1)) = \frac{b}{b+1}\, d(x_1, \partial D).$$

If $x \in \gamma(y_1, x_0)$, then

$$d(x, \partial D) \geqq \frac{1}{b}\, s(\gamma(x_1, x))$$

by (2.1) and hence

$$\begin{aligned} k_D(y_1, x_0) &\leqq b \log\left(\frac{b+1}{b}\, \frac{s(\gamma(x_1, x_0))}{d(x_1, \partial D)}\right) \\ &< b \log\left(\frac{s(\gamma(x_1, x_2))}{d(x_1, \partial D)}\right) + 1 \\ &< b \log a\left(\frac{|x_1 - x_2|}{d(x_1, \partial D)} + 1\right) + 1 \end{aligned}$$

again by (2.1). Now $k_D(x_1, y_1) \leqq b$ by what was proved above, and (2.3) follows from the triangle inequality.

A rectifiable arc $\gamma \subset D$ is said to be a *quasi-hyperbolic geodesic* if

$$k_D(y_1, y_2) = \int_{\gamma(y_1, y_2)} d(x, \partial D)^{-1} ds \tag{2.5}$$

for each pair of points $y_1, y_2 \in \gamma$. Obviously each subarc of a quasi-hyperbolic geodesic is again a geodesic.

Lemma 1. *For each pair of points $x_1, x_2 \in D$ there exists a quasi-hyperbolic geodesic γ with x_1 and x_2 as its end points.*

Proof. Fix $x_1, x_2 \in D$. By definition there exists a sequence of rectifiable arcs γ_j joining x_1 and x_2 in D such that

$$k_D(x_1, x_2) = \lim_{j \to \infty} \int_{\gamma_j} d(x, \partial D)^{-1} ds.$$

Obviously we may assume that

$$c = \sup_j \int_{\gamma_j} d(x, \partial D)^{-1} ds < \infty.$$

If $x \in \gamma_j$, then (1.2) implies that

$$\log \frac{d(x, \partial D)}{d(x_1, \partial D)} \leqq k_D(x_1, x) \leqq \int_{\gamma_j} d(x, \partial D)^{-1} ds,$$

whence

$$d(x, \partial D) \leqq e^c d(x_1, \partial D).$$

Thus

$$s(\gamma_j) \leqq e^c d(x_1, \partial D) \int_{\gamma_j} d(x, \partial D)^{-1} ds \leqq c e^c d(x_1, \partial D) \tag{2.6}$$

and the γ_j have uniformly bounded euclidean length. From the Helly selection principle we obtain a subsequence $\{j_k\}$ and a rectifiable curve γ joining x_1 and x_2 in D such that

$$k_D(x_1, x_2) = \lim_{k \to \infty} \int_{\gamma_{j_k}} d(x, \partial D)^{-1} ds = \int_{\gamma} d(x, \partial D)^{-1} ds. \tag{2.7}$$

(See, for example, pp. 72–75 in [17].) Then (2.7) implies that γ is an arc, and with the triangle inequality we see that (2.5) holds for all $y_1, y_2 \in \gamma$.

Theorem 1 implies that $k_D \leqq c j_D + d$ if D is a uniform domain. We show now that this inequality holds only if D is uniform by establishing the following result.

Theorem 2. *Suppose that γ is a quasi-hyperbolic geodesic in D and suppose that*

$$k_D(y_1, y_2) \leqq c j_D(y_1, y_2) + d \tag{2.8}$$

for all $y_1, y_2 \in \gamma$. Then

$$\begin{cases} s(\gamma(x_1, x_2)) \leqq a\,|x_1 - x_2|, \\ \min_{j=1,2} s(\gamma(x_j, x)) \leqq a d(x, \partial D) \end{cases} \tag{2.9}$$

for each ordered triple of points $x_1, x, x_2 \in \gamma$, where

$$a = 2be^{2b}, \qquad b = \max(8c^2 e^{d/c}, 1).$$

Proof. Fix $x_1, x_2 \in \gamma$. To establish (2.9) we may assume that $\gamma = \gamma(x_1, x_2)$. Set

$$r = \min\left(\sup_{x\in\gamma} d(x,\partial D), 2|x_1 - x_2|\right).$$

We shall consider the cases where

$$r < \max_{j=1,2} d(x_j, \partial D)$$

and where

$$(2.10) \qquad r \geqq \max_{j=1,2} d(x_j, \partial D)$$

separately.

Suppose first that $r < d(x_1, \partial D)$. Then $r = 2|x_1 - x_2|$ and

$$|x_1 - x_2| \leqq \frac{1}{2} d(x_1, \partial D) \leqq d(x, \partial D)$$

for all x on the segment β joining x_1 and x_2. Thus

$$k_D(x_1, x_2) \leqq \int_\beta d(x, \partial D)^{-1} ds \leqq \frac{2|x_1 - x_2|}{d(x_1, \partial D)} \leqq 1,$$

and by (1.2)

$$\frac{1}{e} d(x_1, \partial D) \leqq d(x, \partial D) \leqq e d(x_1, \partial D)$$

for each $x \in \gamma$. These inequalities imply that

$$s(\gamma) \leqq e d(x_1, \partial D) \int_\gamma d(x, \partial D)^{-1} ds \leqq 2e |x_1 - x_2|,$$

that

$$s(\gamma(x_1, x)) \leqq s(\gamma) \leqq e d(x_1, \partial D) \leqq e^2 d(x, \partial D)$$

for each $x \in \gamma$, and (2.9) follows since $a \geqq e^2$. Similarly if $r < d(x_2, \partial D)$, we again obtain (2.9) by reversing the roles of x_1 and x_2 in the above argument.

Suppose next that (2.10) holds. By compactness there exists a point $x_0 \in \gamma$ with

$$r \leqq \sup_{x \in \gamma} d(x, \partial D) = d(x_0, \partial D).$$

Next for $j = 1, 2$ let m_j denote the largest integer for which

$$2^{m_j} d(x_j, \partial D) \leqq r,$$

and let y_j be the first point of $\gamma(x_j, x_0)$ with

$$d(y_j, \partial D) = 2^{m_j} d(x_j, \partial D)$$

as we traverse γ from x_j towards x_0. Obviously

$$d(y_j, \partial D) \leqq r < 2d(y_j, \partial D). \tag{2.11}$$

We show first that

$$\begin{cases} s(\gamma(x_j, y_j)) \leqq bd(y_j, \partial D), \\ s(\gamma(x_j, x)) \leqq be^b d(x, \partial D) \quad \text{for } x \in \gamma(x_j, y_j), \end{cases} \tag{2.12}$$

for $j = 1, 2$. Clearly we need only consider the case where $j = 1$ and $m_1 \geqq 1$. For this choose points $z_1, \cdots, z_{m_1+1} \in \gamma(x_1, y_1)$ so that $z_1 = x_1$ and so that z_j is the first point of $\gamma(x_1, y_1)$ for which

$$d(z_j, \partial D) = 2^{j-1} d(x_1, \partial D) \tag{2.13}$$

as we traverse γ from x_1 towards y_1. Then $z_{m_1+1} = y_1$. Fix j and set

$$t = \frac{s(\gamma(z_j, z_{j+1}))}{d(z_j, \partial D)}.$$

If $x \in \gamma(z_j, z_{j+1})$, then

$$d(x, \partial D) \leqq d(z_{j+1}, \partial D) = 2d(z_j, \partial D),$$

and hence

$$t \leqq 2 \int_{\gamma_j} d(x, \partial D)^{-1} ds = 2k_D(z_j, z_{j+1}), \qquad \gamma_j = \gamma(z_j, z_{j+1}),$$

because γ is a quasi-hyperbolic geodesic. Since

$$j_D(z_j, z_{j+1}) \leqq \log\left(\frac{|z_j - z_{j+1}|}{d(z_j, \partial D)} + 1\right) \leqq \log(t+1),$$

inequality (2.8) implies that

$$k_D(z_j, z_{j+1}) \leqq c \log(e^{d/c}(t+1)) \leqq c(e^{d/c}(t+1))^{1/2}.$$

If $t \geqq 1$, we see from the above inequalities that

$$t \leqq 8c^2 e^{d/c} \leqq b, \tag{2.14}$$

and hence that

$$k_D(z_j, z_{j+1}) \leqq c(2be^{d/c})^{1/2} < b. \tag{2.15}$$

If $t < 1$, then $t < b$ and again we obtain (2.15). We conclude from (1.2) that

$$\begin{cases} s(\gamma(z_j, z_{j+1})) \leqq bd(z_j, \partial D), \\ d(z_{j+1}, \partial D) \leqq e^b d(x, \partial D) \quad \text{for } x \in \gamma(z_j, z_{j+1}), \end{cases} \tag{2.16}$$

for $j = 1, \cdots, m_1$. Hence

$$\begin{aligned} s(\gamma(x_1, y_1)) &= \sum_{j=1}^{m_1} s(\gamma(z_j, z_{j+1})) \leqq b \sum_{j=1}^{m_1} d(z_j, \partial D) \\ &= b(2^{m_1} - 1)d(x_1, \partial D) < bd(y_1, \partial D) \end{aligned}$$

by (2.13) and (2.16). Next if $x \in \gamma(x_1, y_1)$, then $x \in \gamma(z_j, z_{j+1})$ for some j and

$$\begin{aligned} s(\gamma(x_1, x)) &\leqq \sum_{i=1}^{j} s(\gamma(z_i, z_{i+1})) \leqq b \sum_{i=1}^{j} d(z_i, \partial D) \\ &< bd(z_{j+1}, \partial D) \leqq be^b d(x, \partial D) \end{aligned}$$

again by (2.13) and (2.16). This completes the proof of (2.12).

We show next that if $d(y_1, \partial D) \leqq d(y_2, \partial D)$, then

$$\begin{cases} s(\gamma(y_1, y_2)) \leqq be^b d(y_1, \partial D), \\ d(y_2, \partial D) \leqq e^b d(x, \partial D) \quad \text{for } x \in \gamma(y_1, y_2). \end{cases} \tag{2.17}$$

Obviously we may assume that $y_1 \neq y_2$ since otherwise there is nothing to prove.

Suppose first that

$$r = \sup_{x \in \gamma} d(x, \partial D)$$

and set

$$t = \frac{s(\gamma(y_1, y_2))}{d(y_1, \partial D)}.$$

If $x \in \gamma(y_1, y_2)$, then

$$d(x, \partial D) \leqq r < 2d(y_1, \partial D)$$

by (2.11) and we can repeat the proof of (2.16), with z_j replaced by y_1 and z_{j+1} by y_2, to obtain (2.17). Suppose next that $r = 2|x_1 - x_2|$. Then the triangle inequality, (2.11) and (2.12) imply that

$$\begin{aligned} |y_1 - y_2| &\leqq s(\gamma(x_1, y_1)) + s(\gamma(x_2, y_2)) + |x_1 - x_2| \\ &\leqq bd(y_1, \partial D) + bd(y_2, \partial D) + \frac{r}{2} \\ &\leqq 4bd(y_1, \partial D). \end{aligned}$$

Hence $j_D(y_1, y_2) \leqq \log 5b$ and

$$k_D(y_1, y_2) \leqq c \log(5be^{d/c}) \leqq c(5be^{d/c})^{1/2} < b$$

by (2.8). If $x \in \gamma(y_1, y_2)$, then

$$e^{-b} d(y_2, \partial D) \leqq d(x, \partial D) \leqq e^b d(y_1, \partial D)$$

by (1.2),

$$s(\gamma(y_1, y_2)) \leqq e^b d(y_1, \partial D) k_D(y_1, y_2) \leqq be^b d(y_1, \partial D)$$

and again we obtain (2.17).

We now complete the proof of Theorem 2 as follows. By relabeling we may assume that $d(y_1, \partial D) \leqq d(y_2, \partial D)$. Then

$$\begin{aligned} s(\gamma) &= s(\gamma(x_1, y_1)) + s(\gamma(x_2, y_2)) + s(\gamma(y_1, y_2)) \\ &\leqq 2be^b d(y_2, \partial D) \\ &\leqq 4be^b |x_1 - x_2| \end{aligned}$$

by (2.11), (2.12) and (2.17). This establishes the first part of (2.9). Next if $x \in \gamma$, then either $x \in \gamma(x_j, y_j)$ and

$$\min_{j=1,2} s(\gamma(x_j, x)) \leqq s(\gamma(x_j, x)) \leqq be^b d(x, \partial D)$$

by (2.12), or $x \in \gamma(y_1, y_2)$ and

$$\min_{j=1,2} s(\gamma(x_j, x)) \leqq \frac{1}{2} s(\gamma) \leqq be^b d(y_2, \partial D)$$
$$\leqq be^{2b} d(x, \partial D)$$

by (2.17). In each case we obtain the second part of (2.9) and the proof is complete.

Theorem 1, Lemma 1 and Theorem 2 yield the following characterization for uniform domains.

Corollary 1. *A domain D is uniform if and only if there exist constants c and d such that*

$$k_D(x_1, x_2) \leqq c j_D(x_1, x_2) + d$$

for all $x_1, x_2 \in D$.

These results also yield the following information about the quasi-hyperbolic geodesics in uniform domains.

Corollary 2. *If D is a uniform domain, then there exist constants a and b such that*

$$s(\gamma(x_1, x_2)) \leqq a\,|x_1 - x_2|,$$

$$\min_{j=1,2} s(\gamma(x_j, x)) \leqq bd(x, \partial D)$$

for each quasi-hyperbolic geodesic γ *in D and each ordered triple of points* $x_1, x, x_2 \in \gamma$.

Suppose next that ρ_D is a function continuous in D and suppose that there exists a constant m such that

$$\frac{1}{m} d(x, \partial D)^{-1} \leqq \rho_D(x) \leqq m d(x, \partial D)^{-1} \tag{2.18}$$

for all $x \in D$. Next let

$$h_D(x_1, x_2) = \inf_\gamma \int_\gamma \rho_D(x)ds,$$

where the infimum is taken over all rectifiable arcs γ which join x_1 and x_2 in D. Then h_D is a metric in D and

$$\frac{1}{m} h_D(x_1, x_2) \leqq k_D(x_1, x_2) \leqq m h_D(x_1, x_2) \tag{2.19}$$

for all $x_1, x_2 \in D$.

Remark. Theorem 1, Lemma 1, Theorem 2 and hence Corollaries 1 and 2 all hold with k_D replaced by the metric h_D.

Proof. Using (2.19) we see that the conclusion in Theorem 1 takes the form

$$h_D(x_1, x_2) \leqq c j_D(x_1, x_2) + d,$$

where $c = 2mb$ and $d = 2m(b + b \log a + 1)$, while the first half of (1.2) becomes

$$\left| \log \frac{d(x_1, \partial D)}{d(x_2, \partial D)} \right| \leqq m h_D(x_1, x_2). \tag{2.20}$$

The proof of Lemma 1 then follows from (2.18) and (2.20) with the constant ce^c in (2.6) replaced by mce^{mc}. Finally if we carry through the proof of Theorem 2 assuming that

$$h_D(y_1, y_2) \leqq c j_D(y_1, y_2) + d$$

for all y_1, y_2 on an h_D-geodesic γ, we again obtain (2.9) with

$$a = 2mbe^{2mb}, \qquad b = \max(8m^2c^2e^{d/c}, m).$$

3. Quasi-invariance of j_D and k_D

We begin with two results on distance distortion under quasiconformal mappings.

Lemma 2. *There exists a constant a depending only on n with the following property. If f is a K-quasiconformal mapping of D onto D', then*

$$\frac{|f(x_1)-f(x_2)|}{d(f(x_1),\partial D')} \leqq a\left(\frac{|x_1-x_2|}{d(x_1,\partial D)}\right)^{\alpha}, \qquad \alpha = K^{1/(1-n)}, \tag{3.1}$$

for all $x_1, x_2 \in D$ *with*

$$\frac{|x_1-x_2|}{d(x_1,\partial D)} \leqq a^{-1/\alpha}.$$

Proof. By assumption D and D' are proper subdomains of R^n. Then by the n-dimensional analogue of theorem 11 in [5],

$$\frac{|f(x_1)-f(x_2)|}{d(f(x_1),\partial D')} \leqq \theta_K\left(\frac{|x_1-x_2|}{d(x_1,\partial D)}\right) \tag{3.2}$$

for all $x_1, x_2 \in D$ with $|x_1 - x_2| < d(x_1, \partial D)$. (See p. 248 in [3].) Here

$$\theta_K(t) = (\Psi^{-1}(\Phi(1/t)^{\alpha}))^{-1}$$

for $0 < t < 1$, where $\log \Phi(s)$ and $\log \Psi(s)$ denote respectively the conformal moduli of the Grötzsch and Teichmüller ring domains, $R_G(s)$ and $R_T(s)$, in R^n. That is,

$$R_G(s) = R^n - \{x : |x| \leqq 1\} - \{x = ue_1 : s \leqq u < \infty\},$$

for $1 < s < \infty$ and

$$R_T(s) = R^n - \{x = ue_1 : -1 \leqq u \leqq 0, s \leqq u < \infty\}$$

for $0 < s < \infty$, where $e_1 = (1, 0, \cdots, 0)$. It is well known that

$$s \leqq \Phi(s) \leqq \lambda_n s, \qquad \Psi(s) = \Phi((s+1)^{1/2})^2 \tag{3.3}$$

where $4 \leqq \lambda_n < e^n$. (See [2], [3] and [13].) From (3.3) it follows that

$$\theta_K(t) \leqq 2\lambda_n^2 t^{\alpha} \tag{3.4}$$

if $0 < t \leqq (2\lambda_n^2)^{-1/\alpha}$, and hence we obtain (3.1) from (3.2) and (3.4) with $a = 2\lambda_n^2$, $32 \leqq a < 2e^{2n}$.

Lemma 3. *If f is a K-quasiconformal mapping of $\bar{R}^n$ which fixes ∞, then*

$$\frac{|f(x_1)-f(x_2)|}{|f(x_1)-f(x_3)|}+1 \leqq b\left(\frac{|x_1-x_2|}{|x_1-x_3|}+1\right)^{1/\alpha}, \qquad \alpha = K^{1/(1-n)}, \tag{3.5}$$

for all $x_1, x_2, x_3 \in R^n$ where $b = 2a^{1/\alpha}$, and a is the constant in Lemma 2.

Proof. Fix distinct points $x_1, x_2, x_3 \in R^n$ and let $y_j = f(x_j)$, $D = R^n - \{x_2\}$, $D' = R^n - \{y_2\}$. We may assume that

$$\frac{|y_1-y_3|}{|y_1-y_2|} \leqq \frac{2}{b} = a^{-1/\alpha},$$

since otherwise (3.5) would follow trivially. Then

$$|y_1-y_2| = d(y_1, \partial D'), \qquad |x_1-x_2| = d(x_1, \partial D)$$

and we can apply Lemma 2 to f^{-1} to obtain

$$\frac{|x_1-x_3|}{|x_1-x_2|} \leqq a\left(\frac{|y_1-y_3|}{|y_1-y_2|}\right)^{\alpha},$$

which in turn yields (3.5).

From Lemma 2 we obtain the following result on how k_D changes under a quasiconformal mapping.

Theorem 3. *There exists a constant c depending only on n and K with the following property. If f is a K-quasiconformal mapping of D onto D', then*

$$k_{D'}(f(x_1), f(x_2)) \leqq c \max(k_D(x_1,x_2), k_D(x_1,x_2)^{\alpha}), \qquad \alpha = K^{1/(1-n)}, \tag{3.6}$$

for all $x_1, x_2 \in D$.

Proof. Fix $x_1, x_2 \in D$ and suppose first that

$$\frac{|x_1-x_2|}{d(x_1, \partial D)} \leqq (2a)^{-1/\alpha} < 1, \tag{3.7}$$

where a is the constant in Lemma 2. Then

$$\frac{|f(x_1)-f(x_2)|}{d(f(x_1), \partial D')} \leqq a\left(\frac{|x_1-x_2|}{d(x_1, \partial D)}\right)^{\alpha} \leqq \frac{1}{2} \tag{3.8}$$

by Lemma 2 and

$$d(y, \partial D') \geqq \frac{1}{2} d(f(x_1), \partial D')$$

for all y on the segment β joining $f(x_1)$ and $f(x_2)$. Hence

$$k_{D'}(f(x_1), f(x_2)) \leqq \frac{2|f(x_1) - f(x_2)|}{d(f(x_1), \partial D')} \leqq 1. \tag{3.9}$$

Next

$$k_D(x_1, x_2) \geqq \log\left(\frac{|x_1 - x_2|}{d(x_1, \partial D)} + 1\right) \geqq \frac{1}{2} \frac{|x_1 - x_2|}{d(x_1, \partial D)} \tag{3.10}$$

by (1.2) and (3.7), and we obtain

$$k_{D'}(f(x_1), f(x_2)) \leqq 4a k_D(x_1, x_2)^\alpha \tag{3.11}$$

from (3.8), (3.9) and (3.10).

Suppose next that (3.7) does not hold and choose $y_1, \cdots, y_{m+1}$ on the quasi-hyperbolic geodesic joining x_1 and x_2 so that $y_1 = x_1$, $y_{m+1} = x_2$ and

$$\frac{|y_j - y_{j+1}|}{d(y_j, \partial D)} = (2a)^{-1/\alpha}, \qquad \frac{|y_m - y_{m+1}|}{d(y_m, \partial D)} \leqq (2a)^{-1/\alpha}$$

for $j = 1, \cdots, m-1$. Then

$$k_{D'}(f(x_1), f(x_2)) \leqq \sum_{j=1}^{m} k_{D'}(f(y_j), f(y_{j+1})) \leqq m$$

by (3.9) while

$$k_D(x_1, x_2) = \sum_{j=1}^{m} k_D(y_j, y_{j+1}) \geqq \frac{m-1}{2} (2a)^{-1/\alpha}$$

by (3.10). Thus

$$k_{D'}(f(x_1), f(x_2)) \leqq 4(2a)^{1/\alpha} k_D(x_1, x_2) \tag{3.12}$$

since $m \geqq 2$. Inequality (3.6) then follows from (3.11) and (3.12) with $c = 4(2a)^{1/\alpha}$.

We have next the following analogue of Theorem 3 for the function j_D.

Theorem 4. *There exist constants c and d depending only on n and K with the following property. If f is a K-quasiconformal mapping of $\bar{R}^n$ which maps D onto D', then*

$$j_{D'}(f(x_1), f(x_2)) \leqq c j_D(x_1, x_2) + d \tag{3.13}$$

for all $x_1, x_2 \in D$.

Proof. Fix $x_1, x_2 \in D$ and suppose first that f is a Möbius transformation. Choose $x_3 \in \partial D$ and $x_4 \in \bar{R}^n - D$ so that

$$|f(x_1) - f(x_3)| = d(f(x_1), \partial D') \tag{3.14}$$

and $f(x_4) = \infty$. Since f is a Möbius transformation,

$$\frac{|f(x_1) - f(x_2)|}{d(f(x_1), \partial D')} = \frac{|f(x_1) - f(x_2)|}{|f(x_1) - f(x_3)|} = \frac{|x_1 - x_2|}{|x_1 - x_3|} \frac{|x_3 - x_4|}{|x_2 - x_4|}. \tag{3.15}$$

If $x_4 = \infty$, then (3.15) implies that

$$\frac{|f(x_1) - f(x_2)|}{d(f(x_1), \partial D')} = \frac{|x_1 - x_2|}{|x_1 - x_3|} \leqq \frac{|x_1 - x_2|}{d(x_1, \partial D)}$$

since $d(x_1, \partial D) \leqq |x_1 - x_3|$; hence

$$\frac{|f(x_1) - f(x_2)|}{d(f(x_1), \partial D')} + 1 \leqq \left(\frac{|x_1 - x_2|}{d(x_1, \partial D)} + 1\right)\left(\frac{|x_1 - x_2|}{d(x_2, \partial D)} + 1\right). \tag{3.16}$$

If $x_4 \neq \infty$, then

$$|x_3 - x_4| \leqq |x_1 - x_2| + |x_1 - x_3| + |x_2 - x_4|,$$

and (3.15) implies that

$$\frac{|f(x_1) - f(x_2)|}{d(f(x_1), \partial D')} + 1 \leqq \left(\frac{|x_1 - x_2|}{|x_1 - x_3|} + 1\right)\left(\frac{|x_1 - x_2|}{|x_2 - x_4|} + 1\right);$$

hence (3.16) again holds since $d(x_2, \partial D) \leqq |x_2 - x_4|$. We conclude that

$$j_{D'}(f(x_1), f(x_2)) \leqq 2 j_D(x_1, x_2) \tag{3.17}$$

from interchanging the roles of x_1 and x_2 in (3.16), taking logarithms and then adding.

Suppose next that f is K-quasiconformal and fixes ∞, and choose $x_3 \in \partial D$ so that (3.14) holds. Then

$$\frac{|f(x_1)-f(x_2)|}{d(f(x_1),\partial D')}+1 \leqq b\left(\frac{|x_1-x_2|}{d(x_1,\partial D)}+1\right)^{1/\alpha}, \qquad \alpha = K^{1/(1-n)},$$

by Lemma 3, and again as above we obtain

$$j_{D'}(f(x_1),f(x_2)) \leqq \frac{1}{\alpha}\, j_D(x_1,x_2)+\log b. \tag{3.18}$$

For the general case we can write $f = g\circ h$ where h is a Möbius transformation and where g fixes ∞. Then (3.13) follows from (3.17) and (3.18) with $c = 2/\alpha$ and $d = \log b$.

The quasiconformal invariance of uniform domains is now an immediate consequence of Corollary 1 and Theorems 3 and 4. (See theorems 6.2 in [11] and 2.15 in [12].)

Corollary 3. *If D is a uniform domain and if f is a quasiconformal mapping of $\bar{R}^n$ which maps D onto D', then D' is a uniform domain.*

Proof. By Corollary 1 there exist constants c and d such that

$$k_D(x_1,x_2) \leqq c j_D(x_1,x_2)+d$$

for all $x_1, x_2 \in D$. Next by Theorems 3 and 4 there exist constants c_1, c_2, d_2 depending only on n and K such that

$$k_{D'}(y_1,y_2) \leqq c_1(k_D(x_1,x_2)+1),$$

$$j_D(x_1,x_2) \leqq c_2 j_{D'}(y_1,y_2)+d_2$$

for all $y_1, y_2 \in D'$ where $x_j = f^{-1}(y_j)$. Hence

$$k_{D'}(y_1,y_2) \leqq c' j_{D'}(y_1,y_2)+d'$$

for all $y_1, y_2 \in D'$ where $c' = c_1c_2c$, $d' = c_1(cd_2+d+1)$, and D' is uniform by Corollary 1.

Now it is easy to check that if D is a half space in R^n, then D satisfies (1.4) with $c = 2$ and $d = 0$. Hence we obtain the following result.

Corollary 4. *There exist constants a and b depending only on K and n with the following property. If D is the image of a ball or half space under a K-quasiconformal mapping of* $\bar{R}^n$, *then*

$$
(3.19)\qquad \begin{cases} s(\gamma(x_1, x_2)) \leqq a\,|x_1 - x_2|, \\ \min\limits_{j=1,2} s(\gamma(x_j, x)) \leqq b d(x, \partial D) \end{cases}
$$

for each quasi-hyperbolic geodesic γ *in D and each ordered triple of points* $x_1, x, x_2 \in \gamma$. *Moreover when* $n = 2$, (3.19) *also holds for each hyperbolic geodesic* γ *in D and each ordered triple of points* $x_1, x, x_2 \in \gamma$.

Proof. We may assume that D is the image of a half space H under a K-quasiconformal mapping f of $\bar{R}^n$. Then $k_D \leqq cj_D + d$ in D where c and d depend only on K and n, and (3.19) follows for quasi-hyperbolic geodesics from Theorem 2.

When $n = 2$, the density ρ_D for the hyperbolic metric h_D in D satisfies the inequality

$$
\frac{1}{4} d(x, \partial D)^{-1} \leqq \rho_D(x) \leqq d(x, \partial D)^{-1}
$$

by virtue of the Koebe distortion theorem and the Schwarz Lemma. Hence (3.19) holds for hyperbolic geodesics γ in D by the Remark in section 2.

4. Quasiconformally decomposable domains

We use results in the last two sections to obtain a new characterization for uniform domains in R^2.

Theorem 5. *A domain D in* R^2 *is uniform if and only if it is quasiconformally decomposable.*

Proof. Suppose that D is uniform. We want to find a constant K with the following property. For each pair of points $z_1, z_2 \in D$ there exists a subdomain D_0 of D such that $z_1, z_2 \in \bar{D}_0$ and such that ∂D is a K-quasiconformal circle.

Fix $z_1, z_2 \in D$ and let γ denote a quasi-hyperbolic geodesic in D with z_1 and z_2 as its end points. Then by Corollary 2,

$$(4.1)\qquad \begin{cases} s(\gamma(w_1, w_2)) \leqq a_1 |w_1 - w_2| & \text{for all } w_1, w_2 \in \gamma, \\ \min_{j=1,2} s(\gamma(z_j, z)) \leqq b_1 d(z, \partial D) & \text{for all } z \in \gamma, \end{cases}$$

where a_1 and b_1 are constants depending only on D. Next for each ordered quadruple of points $w_1, w_2, w_3, w_4 \in \gamma$ let

$$c(w_1, w_2, w_3, w_4) = \frac{|w_1 - w_2||w_3 - w_4|}{|w_1 - w_3||w_2 - w_4|} + \frac{|w_1 - w_4||w_2 - w_3|}{|w_1 - w_3||w_2 - w_4|}.$$

From (4.1) it follows that

$$\max(|w_1 - w_2|, |w_2 - w_3|) \leqq s(\gamma(w_1, w_3)) \leqq a_1 |w_1 - w_3|,$$

$$\max(|w_2 - w_3|, |w_3 - w_4|) \leqq a_1 |w_2 - w_4|,$$

$$|w_1 - w_4| \leqq |w_1 - w_3| + a_1 |w_2 - w_4|,$$

and hence $c(w_1, w_2, w_3, w_4) \leqq 2a_1^2 + a_1$. By theorem 1 of [16] there exists a K_1-quasiconformal mapping f of $\bar{R}^2$ which fixes ∞ and maps γ onto a segment γ' in the real axis so that $f(z_1) < f(z_2)$; moreover K_1 is a constant which depends only on a_1. Let $u_j = f(z_j)$ and set $c_1 = \max(b_1, 1)$ and

$$D_0' = \{w = u + iv : |v| < (ac_1)^{-K_1} \min(u - u_1, u_2 - u)\},$$

where a is the constant in Lemma 2 when $n = 2$. Then D_0' is a domain which contains $f(z_1)$, $f(z_2)$ in its closure and $\partial D_0'$ is a K_2-quasiconformal circle where K_2 depends only on b_1 and K_1.

Let $D_1 = R^2 - \{z_1, z_2\}$, $D_1' = f(D_1)$ and fix $w = u + iv \in D_0'$. If $w_0 = u$, then

$$|w - w_0| < (ac_1)^{-K_1} d(w_0, \partial D_1')$$

and hence

$$\frac{|z - z_0|}{d(z_0, \partial D_1)} \leqq a \left(\frac{|w - w_0|}{d(w_0, \partial D_1')} \right)^{1/K_1} < \frac{1}{c_1}$$

by Lemma 2 applied to f^{-1}, where $z = f^{-1}(w)$ and $z_0 = f^{-1}(w_0)$. Since

$$d(z_0, \partial D_1) \leqq \min_{j=1,2} s(\gamma(z_j, z_0)) \leqq b_1 d(z_0, \partial D),$$

we conclude that $|z - z_0| < d(z_0, \partial D)$ and hence that $z \in D$. Thus $D_0 = f^{-1}(D_0')$ is a subdomain of D, $z_1, z_2 \in \bar{D}_0$ and ∂D_0 is a K-quasiconformal circle where $K = K_1K_2$ depends only on a_1 and b_1. This completes the proof of the necessity part of Theorem 5.

For the sufficiency part suppose that D is quasiconformally decomposable and fix $z_1, z_2 \in D$. By hypothesis there exists a subdomain D_0 of D such that $z_1, z_2 \in \bar{D}_0$ and such that ∂D_0 is a K-quasiconformal circle, where K depends only on D. With the generalized Riemann mapping theorem [10] and lemma 1 in [18] we obtain a K^2-quasiconformal mapping f of $\bar{R}^2$ which maps D_0 conformally onto the unit disk so that $f(z_1)$ and $f(z_2)$ lie on the real axis. Let β denote the closed segment joining $f(z_1)$ and $f(z_2)$ and let $\gamma = f^{-1}(\beta)$. If w_1, z, w_2 is any ordered triple of points on $\gamma \cap D_0$, then $\gamma(w_1, w_2)$ is a hyperbolic geodesic in D_0 and

$$(4.2) \qquad \begin{cases} s(\gamma(w_1, w_2)) \leqq a\,|w_1 - w_2|, \\ \min_{j=1,2} s(\gamma(w_j, z)) \leqq b d(z, \partial D_0) \leqq b d(z, \partial D) \end{cases}$$

by Corollary 4, where a and b are constants which depend only on K and hence on D. If we now let $w_1 \to z_1$ and $w_2 \to z_2$ along γ, we obtain (4.2) with z_j in place of w_j. Thus D is uniform and the proof of Theorem 5 is complete.

Theorem 5 yields a second proof of Corollary 3 for the case when $n = 2$, since the image of a quasiconformally decomposable domain under a quasiconformal mapping of $\bar{R}^2$ is again clearly quasiconformally decomposable.

Theorem 5 also yields a new proof of the main injectivity properties of uniform domains in R^2. We require first the following result essentially due to Duren, Shapiro and Shields [4].

Lemma 4. *If g is analytic in a domain D in R^2, then*

$$(4.3) \qquad \sup_{z \in D} |g'(z)|\, d(z, \partial D)^2 \leqq 4 \sup_{z \in D} |g(z)|\, d(z, \partial D).$$

Proof. We may clearly assume that

$$\sup_{z \in D} |g(z)|\, d(z, \partial D) = c < \infty.$$

Fix $z \in D$ and let $r = \frac{1}{2} d(z, \partial D)$. Then the Cauchy integral formula implies that

$$|g'(z)| \leqq \frac{1}{r} \sup_{|\zeta - z| = r} |g(\zeta)| \leqq \frac{4c}{d(z, \partial D)^2},$$

and we obtain (4.3).

We have next the following extension, due to Martio and Sarvas [12], of an important and seminal result of Nehari [14].

Theorem 6. *If D is a uniform domain in R^2, then there exist positive constants a and b with the following property. If f is analytic and locally univalent in D and if either*

$$\sup_{z \in D} |S_f(z)|\, d(z, \partial D)^2 \leqq a \tag{4.4}$$

or

$$\sup_{z \in D} \left| \frac{f''(z)}{f'(z)} \right| d(z, \partial D) \leqq b, \tag{4.5}$$

then f is univalent in D.

Here S_f denotes the Schwarzian derivative of f,

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Proof. Suppose first that ∂D is a K-quasiconformal circle. Then by a theorem of Ahlfors ([1] or [9]) there exists a positive constant a depending only on K such that each f satisfying (4.4) must be univalent in $\bar{D}$. Choose $b > 0$ so that $4b + b^2/2 = a$. If f satisfies (4.5), then Lemma 4 applied to $g = f''/f'$ implies that (4.4) holds and hence f must again be univalent in $\bar{D}$.

For the general case fix $z_1, z_2 \in D$ with $z_1 \neq z_2$. By Theorem 5 there exists a subdomain D_0 of D such that $z_1, z_2 \in \bar{D}_0$ and such that ∂D_0 is a K-quasiconformal circle where K depends only on D. Choose a and b corresponding to K as above and suppose that f satisfies the hypotheses of Theorem 6. Then f satisfies the same hypotheses with D replaced by D_0, $f(z_1) \neq f(z_2)$ by what was proved above and hence f is univalent in D.

5. An Example

We say that a domain D in R^2 satisfies the *Schwarzian univalence criterion* if there exists a positive constant a with the following property. If f is analytic and locally univalent in D and if

$$\sup_{z \in D} |S_f(z)|\, d(z, \partial D)^2 \leqq a,$$

then f is univalent in D.

We then have the following characterization for finitely connected uniform plane domains.

Corollary 5. *A finitely connected domain D in R^2 is uniform if and only if it satisfies the Schwarzian univalence criterion.*

Proof. If D is uniform, then D satisfies the Schwarzian univalence criterion by Theorem 6. Conversely if D satisfies the Schwarzian univalence criterion, then D is quasiconformally decomposable by theorem 5 in [15] and hence uniform by Theorem 5 of the present paper.

It is natural to ask if the above characterization holds when D is an infinitely connected plane domain. We present here an example to show that this is not the case. We require first the following result on removable singularities.

Lemma 5. *Suppose that $z_0 \in D \subset R^2$ and that f is analytic and locally univalent in $D - \{z_0\}$. If*

$$\limsup_{z \to z_0} |S_f(z)|\,|z - z_0|^2 < \infty, \tag{5.1}$$

then f has a meromorphic extension to D. If

$$\limsup_{z \to z_0} |S_f(z)|\,|z - z_0|^2 < \frac{3}{2}, \tag{5.2}$$

then f is locally univalent in D.

Proof. It is sufficient to consider the case where $z_0 = 0$ and D is the disk $\{z : |z| < r\}$.

Since $f' \neq 0$, S_f is analytic in $D - \{0\}$. Thus S_f has a meromorphic extension to D with at most a pole of order 2 at $z = 0$ by (5.1), and $z = 0$ is a regular singular point for the differential equation

$$w'' + \frac{1}{2} S_f w = 0, \qquad w = w(z). \tag{5.3}$$

The indicial equation for (5.3) is $\rho^2 - \rho + q = 0$ where

$$q = \frac{1}{2} \lim_{z \to 0} z^2 S_f(z).$$

Let ρ_1, ρ_2 be the roots of this equation numbered so that $\mathrm{Re}(\rho_1) \leqq \mathrm{Re}(\rho_2)$. Then

$$\rho_1 + \rho_2 = 1, \qquad |\rho_1 \rho_2| = |q|. \tag{5.4}$$

Next let D_1 be the slit disk

$$D_1 = D - \{z = t : -r < t \leqq 0\}.$$

By Fuchs' theorem we can find two linearly independent solutions w_1 and w_2 of (5.3) in D_1,

$$w_1(z) = z^{\rho_1} g_1(z),$$

$$w_2(z) = z^{\rho_2} g_2(z) + a w_1(z) \log z,$$

where g_1 and g_2 are analytic in D with $g_1(0) = g_2(0) = 1$ and where a is a constant which is zero if $\rho_2 \not\equiv \rho_1 \pmod 1$ and nonzero if $\rho_2 = \rho_1$. (See, for example, theorem 5.3.1 in [7].) By replacing r by a smaller constant we may assume that $g_1 \neq 0$ in D. Then $h = w_2/w_1$ is analytic with $S_h = S_f$ in D_1 and we can find a Möbius transformation T such that

$$T(f(z)) = h(z) = z^{\rho_2 - \rho_1} g(z) + a \log z \tag{5.5}$$

in D_1, where $g = g_2/g_1$ is analytic in D.

Now (5.5) implies that h has a meromorphic extension to $D - \{0\}$. From this it follows first that $\rho_2 - \rho_1$ is a nonnegative integer n and next that $a = 0$. Thus $h(z) = z^n g(z)$ has an analytic extension to D and $f = T^{-1} \circ h$ is meromorphic in D. Next if (5.2) holds, then

$$n^2 = (\rho_1 + \rho_2)^2 - 4\rho_1\rho_2 \leqq 1 + 4|q| < 4, \qquad n = 1,$$

by (5.4), h has a simple zero at $z = 0$ and f is locally univalent at $z = 0$ and hence in D.

Remark. The function $f(z) = z^2$ with $S_f(z) = -\frac{3}{2} z^{-2}$ shows that the constant in (5.2) cannot be improved.

Theorem 7. *There exists a domain D in R^2 which satisfies the Schwarzian univalence criterion and which is not uniform.*

Proof. Let Q denote the open square

$$Q = \{z = x + iy : |x| < 1, |y| < 1\},$$

and for $j = 1, 2, \cdots$, let

$$\alpha_j = \{z \in Q : d(z, \partial Q) = r_j\}, \qquad r_j = 2^{-j},$$

$$\beta_j = \left\{z \in Q : d(z, \partial Q) = \frac{3}{4} r_j\right\}.$$

Next for each j let B_j denote the set of points in β_j whose coordinates are multiples of $\frac{1}{4} r_j^2$. We shall show that the domain

$$D = Q - \bigcup_{j=1}^{\infty} B_j$$

has the desired properties.

Since ∂Q is a quasiconformal circle, there exists a positive constant c with the following property. If f is meromorphic and locally univalent in Q and if

$$\sup_{z \in Q} |S_f(z)| \, d(z, \partial Q)^2 \leqq c, \tag{5.6}$$

then f is univalent in Q. (See [1] or [9].) Next let $a = \min(c/64, 1)$ and suppose that f is analytic and locally univalent in D with

$$\sup_{z \in D} |S_f(z)| \, d(z, \partial D)^2 \leqq a. \tag{5.7}$$

Then (5.2) holds for each $z_0 \in \partial D \cap Q$, and Lemma 5 implies that f has an extension which is meromorphic and locally univalent in Q. Fix $z_1 \in Q$ and choose j so that

$$r_j < d(z_1, \partial Q) \leqq 2 r_j. \tag{5.8}$$

If $z \in \alpha_j$ then $d(z, \partial D) \geqq \frac{1}{4} r_j$ and by (5.7)

$$|S_f(z)| \leqq 16 a r_j^{-2}. \tag{5.9}$$

The maximum principle, (5.8) and (5.9) then yield

$$|S_f(z_1)| \leqq 16 a r_j^{-2} \leqq c \, d(z_1, \partial Q)^{-2},$$

and we conclude that f is univalent in Q and hence in D. Thus D satisfies the Schwarzian univalence criterion.

Finally suppose that D satisfies the second part of (1.1), fix j so that $br_j < 1$ and choose $z_1 \in \alpha_j \cap D$ and $z_2 \in \alpha_{j+1} \cap D$. By hypothesis there exists a rectifiable arc γ joining z_1 and z_2 in D so that

$$\min_{j=1,2} s(\gamma(z_j, z)) \leqq bd(z, \partial D) \tag{5.10}$$

for all $z \in \gamma$. Let z be the point where γ meets β_j. Then

$$\frac{1}{4} r_j \leqq \min_{j=1,2} |z_j - z|, \quad d(z, \partial D) \leqq \frac{1}{4} r_j^2$$

and with (5.10) we obtain $1 \leqq br_j$ contradicting the way j was chosen. Thus D is not a uniform domain and the proof of Theorem 7 is complete.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109 USA

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