Some Properties of f''/f' and the Poincaré Metric

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1. Introduction. Let B denote the unit disc and S the class of functions f analytic and univalent in B with the normalization $f(0) = 0, f'(0) = 1$. If $f \in S$, then

$$\left| z\frac{f''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2} \right| \leq \frac{4|z|}{1-|z|^2}, \tag{1}$$

and hence

$$\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{6}{1-|z|^2}. \tag{2}$$

See, for example [15, page 21]. These inequalities are sharp for the Koebe function

$$k(z) = \frac{z}{(1-z)^2}, \qquad \frac{k''(z)}{k'(z)} = \frac{2z+4}{1-z^2}.$$

Although (1) and (2) are stated for the class S, they are valid for any univalent function in B since the expression f''/f' is unchanged if f is replaced by $af + b$, $a, b \in \mathbf{C}$, $a \neq 0$. In particular if f is univalent in B then from (1) we have the sharp inequality

$$\left| \frac{f''(0)}{f'(0)} \right| \leq 4. \tag{3}$$

These estimates can be improved if it is known, for example, that $f(B)$ is convex [1, page 5].

In this paper we generalize (2) to an arbitrary simply connected domain and then investigate to what extent this type of inequality holds in a multiply connected domain. We then give an estimate for the gradient of the logarithm of the Poincaré metric of a domain D in terms of the euclidean distance to ∂D. In the simply connected case the estimate follows from a simple but useful connection between this gradient and the operator f''/f' while in the multiply connected case the proof is based on some properties of hyperbolic curvature. The final section is concerned

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with the relationship between these estimates and functions of bounded mean oscillation. A common theme is that of characterizing those domains for which there is a lower bound for the Poincaré metric in terms of the distance to the boundary.

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2. Simply connected domains. Let $D \subset \mathbf{C}$ be a simply connected domain with at least two boundary points and let f be a conformal mapping of B onto D. The hyperbolic or Poincaré metric of D is defined by

$$\lambda_D(f(z))\,|f'(z)| = \lambda_B(z) = \frac{1}{1-|z|^2}, \qquad z \in B. \tag{4}$$

This definition is independent of the choice of conformal mapping and because of this convenient choices are available. Namely, let $w \in D$ and choose the conformal mapping so that $f(0) = w$. Then,

$$\lambda_D(w) = \frac{1}{|f'(0)|}. \tag{5}$$

There is always some question as to the normalization of λ_D. We have defined it to have curvature -4, i.e. to satisfy the equation

$$\Delta \log \lambda_D = 4\lambda_D^2, \tag{6}$$

while it is sometimes preferable to use $2\lambda_D$, which has curvature -1, as in [1, Chapter 1].

At this point the only fact we require about the Poincaré metric is its conformal invariance. If f is a conformal mapping of a domain G onto D then

$$\lambda_D(f(z))\,|f'(z)| = \lambda_G(z), \qquad z \in G. \tag{7}$$

This follows easily from (4) and (5).

Our first result is a generalization of the inequality (2) to any simply connected domain in $\mathbf{C}$ (having at least two boundary points).

Theorem 1. *If $D \subset \mathbf{C}$ is simply connected and if f is analytic and univalent in D then*

$$\left|\frac{f''(z)}{f'(z)}\right| \le 8\lambda_D(z)$$

for all $z \in D$. The inequality is sharp.

Proof. Let $z \in D$ and choose a conformal mapping g of B onto D with $g(0) = z$. For the moment let us write $T_f = f''/f'$. Then $f \circ g$ is univalent in B and a simple computation shows that

$$T_{f\circ g} = (T_f \circ g)g' + T_g. \tag{8}$$

Thus from (3)

$$|g'(0)|\,|T_f(z)| \le |T_{f\circ g}(0)| + |T_g(0)| \le 8$$

and so by (5)

$$\left|\frac{f''(z)}{f'(z)}\right| \le \frac{8}{|g'(0)|} = 8\lambda_D(z)$$

as required.

To show that this is sharp let $D = \mathbf{C}\backslash[0, -\infty)$ and $f(z) = 1/z$. Then f is analytic and univalent in D with $f''(z)/f'(z) = -2/z$. The Poincaré metric of D is determined from (7) by mapping the right half-plane H conformally onto D via $z = w^2$ and using $\lambda_H(w) = (2 \operatorname{Re} w)^{-1}$. We find that along the positive real axis $\lambda_D(x) = (4x)^{-1}$ and so

$$\lambda_D(x)^{-1}\left|\frac{f''(x)}{f'(x)}\right| = 8$$

for all $x > 0$.

Remark. If f is analytic and locally univalent then the Schwarzian derivative of f is defined to be

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

and, among its many properties, it exhibits a transformation law very similar to (8). That is, whenever the composition makes sense,

$$S_{f\circ g} = (S_f \circ g)(g')^2 + S_g.$$

O. Lehto [12] used this to prove that $|S_f(z)| \le 12\lambda_D(z)^2$ whenever f is univalent in a simply connected domain D. The constant 12 is sharp.

3. Multiply connected domains. Any domain $D \subset \mathbf{C}$ with at least two boundary points carries a Poincaré metric. Since the universal covering surface of such a domain is conformally equivalent to the unit disc, there is a locally univalent, analytic covering map p mapping B onto D, and in terms of p

$$\lambda_D(p(z))\,|p'(z)| = \lambda_B(z), \qquad z \in B.$$

Once again this is independent of the choice of p in the sense that it continues to hold if p is replaced by $p \circ h$ where h is a Möbius transformation of the disc onto itself. Just as in the simply connected case λ_D satisfies (6) and (7).

A statement of the type in Theorem 1 does not hold for an arbitrary domain. In fact let $B^x = B\backslash\{0\}$ and $f(z) = 1/z$. Then, as is shown in [1, page 17],

$$\lambda_{B^x}(z) = \frac{1}{2} \frac{1}{|z| \log \frac{1}{|z|}}$$

and we find

$$\sup_{z \in B^x} \lambda_{B^x}(z) \left| \frac{f''(z)}{f'(z)} \right| = \infty.$$

(Note that in [1] the curvature of λ_{B^x} is -1.)

On a more positive note we can give a simple necessary and sufficient condition for a result such as Theorem 1 to hold for a multiply connected domain.

Theorem 2. *Let $D \subset \mathbf{C}$ have at least two boundary points. There exists a constant a such that*

$$\left| \frac{f''(z)}{f'(z)} \right| \leq a\, \lambda_D(z), \qquad z \in D, \tag{9}$$

for all analytic and univalent functions in D if and only if there exists a positive constant c such that

$$\lambda_D(z) \geq \frac{c}{d(z, \partial D)}, \qquad z \in D. \tag{10}$$

Here $d(z, \partial D)$ denotes the euclidean distance to the boundary.

We first need a lemma. See also [13].

Lemma 1. *If D is a proper subdomain of $\mathbf{C}$ and if f is analytic and univalent in D then*

$$\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{4}{d(z, \partial D)},$$

for all $z \in D$. The inequality is sharp.

Proof. Fix $z_0 \in D$ and let $d_0 = d(z_0, \partial D)$. The function $F(z) = f(d_0 z + z_0)$ is univalent in B, whence from (3)

$$\left| \frac{f''(z_0)}{f'(z_0)} \right| = \frac{1}{d_0} \left| \frac{F''(0)}{F'(0)} \right| \leq \frac{4}{d_0} = \frac{4}{d(z_0, \partial D)}.$$

The choice $D = B$ and $f(z) = k(z)$, the Koebe function, shows that the inequality is sharp.

Proof of Theorem 2. If (10) holds and if f is analytic and univalent in D then for all $z \in D$

$$\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{4}{c} \lambda_D(z)$$

by Lemma 1 and we obtain (9) with $a = 4/c$.

Conversely, suppose that (9) holds. Fix $z_0 \in D$ and choose $\zeta_0 \in \partial D$ with $|z_0 - \zeta_0| = d(z_0,\partial D)$. The function

$$f(z) = \frac{1}{z - \zeta_0}$$

is analytic and univalent in D with

$$\frac{f''(z)}{f'(z)} = -\frac{2}{z - \zeta_0}.$$

Therefore by assumption

$$\frac{2}{d(z_0,\partial D)} = \left|\frac{f''(z_0)}{f'(z_0)}\right| \leq a\lambda_D(z_0).$$

Since z_0 was arbitrary we obtain (10) with $c = 2/a$.

We remark that this value of c is correct when D is simply connected by Theorem 1 and (16) in Section 4.

In contrast, the situation for the Schwarzian derivative is much different. A. F. Beardon and F. W. Gehring [4] have shown that $|S_f(z)| \leq 12\lambda_D(z)^2$ for f analytic and univalent in any domain $D \subset \mathbf{C}$. The constant 12 is again sharp. In terms of the distance to the boundary, Gehring [6] has proved the sharp inequality $|S_f(z)| \leq 6d(z,\partial D)^{-2}$.

4. Estimates for the Poincaré metric and hyperbolic curvature. For a differentiable function u we shall use the familiar operator

$$u_z = \frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right), \qquad z = x + iy.$$

If $w = f(z)$ is a conformal mapping of a domain G onto a domain D then from (7)

$$\log \lambda_D \circ f + \log |f'| = \log \lambda_G. \tag{11}$$

Differentiating with respect to z and using the chain rule leads to

$$((\log \lambda_D)_w \circ f)f' + \frac{1}{2}\frac{f''}{f'} = (\log \lambda_G)_z. \tag{12}$$

This relation also holds in the case when $G = B$ and f is analytic covering map onto a multiply connected domain. We record the formula

$$\frac{\partial}{\partial z} \log \lambda_B(z) = \frac{\bar{z}}{1 - |z|^2} = \bar{z}\lambda_B(z). \tag{13}$$

Our first application of (12) is an estimate for $|\nabla \log \lambda_D| = 2|(\log \lambda_D)_z|$ when D is simply connected.

Theorem 3. *If $D \subset \mathbf{C}$ is simply connected then*

$$|\nabla \log \lambda_D(z)| \leq 4\lambda_D(z) \tag{14}$$

$$|\nabla \log \lambda_D(z)| \leq \frac{4}{3}\frac{1}{d(z,\partial D)} \tag{15}$$

for all $z \in D$. Both inequalities are sharp.

These inequalities are just a different way of writing known results for univalent functions, but in this form they admit generalizations to multiply connected domains and (15) has a purely geometric interpretation.

Proof. Fix $z \in D$ and choose a conformal mapping f of B onto D with $f(0) = z$. Then from (12) and (13)

$$|\nabla \log \lambda_D(z)| = \frac{1}{|f'(0)|}\left|\frac{f''(0)}{f'(0)}\right|,$$

hence (14) follows immediately from (3) and (5). Equality holds at $z = 0$ for $D = k(B)$, $k(z)$ the Koebe function.

Next observe that the quantity

$$\frac{d(f(0),\partial D)}{|f'(0)|}\left|\frac{f''(0)}{f'(0)}\right|$$

is invariant under similarity transformations and so the sharp bound for $d(z,\partial D)|\nabla \log \lambda_D(z)|$ in (15) is equivalent to the extremal problem.

$$2 \max_f \,(d(0,\partial D)|a_2|)$$

where $f(z) = z + a_2z^2 + \ldots$ is in the class S and $D = f(B)$. The solution to this problem was found by E. Netanyahu in [14] to be precisely 4/3. His extremal mapping f gives equality in (15) at $z = 0$ when $D = f(B)$ is the domain

$$D = \mathbf{C}\backslash([-r_0,-\infty) \cup \{-r_0e^{i\theta} : |\theta| \leq \theta_0\})$$

where $\theta_0 = 2\tan^{-1}(2\sqrt{2})$ and $r_0 = 4/9$.

The extremal cases in Theorem 3 are quite different and neither inequality implies the other. This is true despite the sharp relations between the Poincaré metric and the distance to the boundary. Namely

$$\frac{1}{4}\frac{1}{d(z,\partial D)} \leq \lambda_D(z) \leq \frac{1}{d(z,\partial D)}, \qquad z \in D. \tag{16}$$

The right-hand inequality holds for any domain and follows from the monotonicity of the Poincaré metric; if $D_1 \subset D$ then $\lambda_{D_1}(z) \geq \lambda_D(z)$, $z \in D_1$. The left-hand inequality holds for any simply connected domain and is implied by (actually equivalent to) the Koebe 1/4-Theorem. Both inequalities are sharp. See, for example, [11, page 45].

Theorem 2 and the example preceding it show that one cannot expect a lower bound of the form $\lambda_D(z) \geq cd(z, \partial D)^{-1}$ to hold in general. Indeed, several analytic and geometric consequences of such a condition have been studied recently by A. F. Beardon and Ch. Pommerenke in [2], [3] and [15]. This condition will occur several times in the sequel, first in extending (14) to a multiply connected domain and again in connection with necessary and sufficient conditions for a function to be of bounded mean oscillation.

To give (15) a geometric interpretation requires some basic properties of hyperbolic curvature, [5, page 22]. Let γ be a C^2 curve in the unit disc having a parameterization $z = z(t)$, $z'(t) \neq 0$. Let κ_e and κ_h denote the euclidean and hyperbolic curvature of γ. If γ passes through the origin then $\kappa_e(0) = \kappa_h(0)$. To find a relation between κ_e and κ_h at any point $z_0 \in \gamma$ we use the invariance of hyperbolic curvature under a conformal mapping f and the formula for the change of euclidean curvature under f as given in [7, Lemma 4];

$$\kappa_e^*(f(z))|f'(z)| = \kappa_e(z) + \operatorname{Im}\left(\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}\right). \tag{17}$$

Here κ_e^* denotes the euclidean curvature of $\gamma^* = f(\gamma)$. Taking $z(t_0) = z_0 \in \gamma$ and f to be the Möbius transformation

$$f(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$$

we find that

$$f'(z_0) = \frac{1}{1 - |z_0|^2} = \lambda_B(z_0), \qquad \frac{f''(z_0)}{f'(z_0)} = \frac{2\bar{z}_0}{1 - |z_0|^2} = 2\frac{\partial}{\partial z}\log \lambda_B(z_0)$$

and on substituting into (17),

$$\begin{aligned}\kappa_h(z_0)\lambda_B(z_0) &= \kappa_h^*(0)\lambda_B(z_0) = \kappa_e^*(0)\lambda_B(z_0)\\ &= \kappa_e(z_0) + \operatorname{Im}\left(2\frac{\partial}{\partial z}\log \lambda_B(z_0)\frac{z'(t_0)}{|z'(t_0)|}\right).\end{aligned} \tag{18}$$

Since $z'(t_0)/|z'(t_0)|$ is the unit tangent to γ at z_0, this can be written more concisely as

$$\kappa_e = \lambda_B\kappa_h + \frac{\partial}{\partial n}\log \lambda_B \tag{19}$$

where $\partial/\partial n$ is the derivative in the normal direction, $+\pi/2$ from $z'/|z'|$. In [10, page 119] V. Jørgensen proved that this relationship between euclidean and hyperbolic curvature holds in any domain D with λ_D replacing λ_B in (19). Deriving (19) first for the disc as above also gives the general result directly from (12) and (17) by taking f to be an analytic covering map of B onto D.

In any domain a hyperbolic geodesic has zero hyperbolic curvature and between any two points there is always a hyperbolic geodesic of smallest hyperbolic length.

Based on an inequality for the Poincaré metric Jørgensen [10, page 122] proved the following.

Lemma 2. *Let γ be a hyperbolic geodesic in a domain $D \subset \mathbf{C}$. Let C_z be the euclidean circle of curvature at a point $z \in \gamma$ and let D_z be the disc bounded by C_z. Then $\overline{D}_z \cap \partial D \neq \emptyset$.*

The lemma is stated to include the case when the euclidean curvature of γ is zero at z and D_z is a half-plane. We shall not need to consider this situation.

If D is simply connected and γ is a hyperbolic geodesic in D, then by (15) and (19) for λ_D,

$$\kappa_e(z) = \frac{\partial}{\partial n} \log \lambda_D(z) \leq |\nabla \log \lambda_D(z)| \leq \frac{4}{3} \frac{1}{d(z,\partial D)}$$

for $z \in \gamma$ and equality can occur. At points where $\kappa_e(z) \neq 0$, $\kappa_e(z)^{-1} \geq (3/4)d(z,\partial D)$; thus, the euclidean circle of curvature to γ at z actually protrudes quite far over ∂D. Hence (15) is a sharp form of Jørgensen's result in the simply connected case. In this connection see also [19].

Lemma 2 also provides a generalization of (15) to multiply connected domains.

Theorem 4. *If $D \subset \mathbf{C}$ is any domain then*

$$|\nabla \log \lambda_D(z)| \leq \frac{2}{d(z,\partial D)} \tag{20}$$

for all $z \in D$.

Proof. Let $z \in D$ and consider the vector $v = \nabla \log \lambda_D(z)$. We may assume that $v \neq 0$ since otherwise the inequality (20) is trivial. Let γ be a hyperbolic geodesic through z with $n = v/|v|$ as the normal vector at z so that

$$\kappa_e(z) = \frac{\partial}{\partial n} \log \lambda_D(z) = |\nabla \log \lambda_D(z)| \neq 0.$$

Then from Lemma 2, $2/\kappa_e(z) \geq d(z,\partial D)$ and the theorem is proved.

It is an open question whether the constant 2 in (20) is sharp, but the following example shows that 2 cannot in general be replaced by 4/3. Let D be the image of $B^x = B\backslash\{0\}$ under $f(z) = (1/2)(z + z^{-1})$. Let $z_0 = i/e$ and $w_0 = f(z_0) = -(i/2)(e - e^{-1})$. Then $(\log \lambda_{B^x})_z(z_0) = 0$ and we calculate from (12) that

$$d(w_0,\partial D)|\nabla \log \lambda_D(w_0)| = \frac{2e^2(e^2 - 1)}{(e^2 + 1)^2} = 1.3416\ldots.$$

An estimate such as (14) does not hold for an arbitrary multiply connected domain D; once again the punctured disc provides a counterexample. A sufficient condition for such an inequality is $\lambda_D(z) \geq cd(z,\partial D)^{-1}$ for then we obtain $|\nabla \log \lambda_D(z)| \leq (2/c)\lambda_D(z)$ by Theorem 4. This condition is also necessary.

Theorem 5. *If* $|\nabla \log \lambda_D(z)| \leq (1/c)\lambda_D(z)$, $z \in D$, *then* $\lambda_D(z) \geq c d(z,\partial D)^{-1}$, $z \in D$.

Proof. Let $z_0 \in D$ and choose $\zeta_0 \in \partial D$ with $|z_0 - \zeta_0| = d(z_0,\partial D)$. Let z be a point on the line segment from z_0 to ζ_0 and denote by β the part of that segment from z_0 to z. Then with arc-length as a parameter for β,

$$\begin{aligned}\left|\frac{1}{\lambda_D(z)} - \frac{1}{\lambda_D(z_0)}\right| &= \left|\int_\beta d\left(\frac{1}{\lambda_D}\right)\right| \\ &= \left|\int_\beta \frac{1}{\lambda_D}\, d(\log \lambda_D)\right| \\ &\leq \int_\beta \frac{1}{\lambda_D} |\nabla \log \lambda_D| ds \\ &\leq \frac{1}{c}|z - z_0| \leq \frac{1}{c} d(z_0,\partial D).\end{aligned}$$

If we now let $z \to \zeta_0$ along the segment then $\lambda_D(z) \to \infty$ and we obtain $\lambda_D(z_0) \geq c d(z_0,\partial D)^{-1}$.

Comparing (14) and (16) we see that this is a sharp result when D is simply connected.

As a corollary of Theorems 2, 4 and 5 we can easily deduce the conformal invariance of the class of domains satisfying $\inf_{z\in D} d(z,\partial D)\lambda_D(z) > 0$.

Corollary 1. *If* $G \subset \mathbf{C}$ *is a domain satisfying* $\lambda_G(z) \geq c d(z,\partial G)^{-1}$, $z \in G$ *and if* $w = f(z)$ *is a conformal mapping of* G *onto* D *then* $\lambda_D(w) \geq (c/6)\, d(w,\partial D)^{-1}$, $w \in D$.

Proof. If $\lambda_G(z) \geq c d(z,\partial G)^{-1}$ then $|f''(z)/f'(z)| \leq (4/c)\lambda_G(z)$ by Theorem 2 and $|\nabla \log \lambda_G(z)| \leq (2/c)\lambda_G(z)$ by Theorem 4. Thus from (12)

$$\begin{aligned}|\nabla \log \lambda_D(w)| &\leq \frac{1}{|f'(z)|}\left(|\nabla \log \lambda_G(z)| + \left|\frac{f''(z)}{f'(z)}\right|\right) \\ &\leq \frac{6}{c}\frac{1}{|f'(z)|}\lambda_G(z) \\ &= \frac{6}{c}\lambda_D(w).\end{aligned}$$

Now from Theorem 5 we conclude that $\lambda_D(w) \geq (c/6)d(w,\partial D)^{-1}$.

The conformal invariance expressed in this corollary can also be obtained from [16, Corollary 2] which we shall reformulate as Lemma 3 in Section 5 for use in another context. The advantage of the present proof is in the simple and explicit dependence on the constant c. See also [16, Proposition 1].

5. Connections with BMO. The estimates in Section 4 are related to necessary and sufficient conditions for a function to be of bounded mean oscillation. That is, a real-valued, locally integrable function u in a domain $D \subset \mathbf{C}$ is said to be of *bounded mean oscillation in D,* written $u \in \mathrm{BMO}(D)$, if

$$\|u\|_{*,D} = \sup_{\Delta} \frac{1}{m(\Delta)} \int_{\Delta} |u - u_{\Delta}| dm < \infty$$

where the supremum is taken over all discs $\Delta \subset D$, dm is two-dimensional Lebesgue measure and

$$u_{\Delta} = \frac{1}{m(\Delta)} \int_{\Delta} u \, dm$$

is the average of u over Δ. With this norm BMO(D) is a Banach space modulo constants.

If D is itself a disc it is not necessary to take the supremum over all discs contained in D. Specifically, if $k > 0$ and if

$$\frac{1}{m(\Delta)} \int_{\Delta} |u - u_{\Delta}| \, dm \leq b$$

for all discs $\Delta \subset D$ satisfying

$$\text{diam } \Delta \leq k d(\Delta, \partial D), \tag{21}$$

then $u \in \mathrm{BMO}(D)$ with $\|u\|_{*,D} \leq b'$, $b' = b'(b,k)$. Here diam Δ is the euclidean diameter of Δ. See [18, Chapter 1].

This criterion will be used in the proof of Theorem 7 in this section. We observe now for use at that time that (21) is also a statement concerning the hyperbolic diameter of Δ. In particular take the case when $D = B$, the unit disc, and let $h(z_1, z_2)$ denote the hyperbolic distance between z_1 and z_2 in B. If Δ is the disc

$$\Delta = \{z : h(z, z_1) < r\}$$

then the euclidean diameter and distance to ∂B of Δ are easily found to be

$$\text{diam } \Delta = 2s \frac{1 - |z_1|^2}{1 - (s|z_1|)^2}, \qquad d(\Delta, \partial B) = \frac{(1 - s)(1 - |z_1|)}{1 + s|z_1|}$$

where $s = \tanh r$. Based on this, straightforward estimates show that

$$\text{diam } \Delta \leq \frac{4s}{(1 - s)^2} d(\Delta, \partial B), \tag{22}$$

and if Δ satisfies (21) then a bound for the hyperbolic diameter of Δ is $2r \leq \log(1 + k)$. Thus, the collection of discs satisfying (21) for a fixed k have uniformly bounded hyperbolic diameters.

In the same connection and also for later use we reformulate [16, Corollary 2]. It gives still another characterization of domains satisfying $\lambda_D(z) \geq c d(z, \partial D)^{-1}$.

Lemma 3. *Let p be an analytic covering map of B onto D. Then* $\inf_{z\in D} d(z,\partial D)\lambda_D(z) \geq c > 0$ *if and only if there is a number $r = r(c) > 0$ such that p is univalent in each disc of hyperbolic radius r.*

Returning to more general considerations, in [8] F. John noted a simple sufficient condition for a differentiable function to be of bounded mean oscillation.

Lemma 4. *If $u : D \to \mathbf{R}$ is differentiable and if* $\sup_{z\in D} d(z,\partial D)|\nabla u(z)| < \infty$ *then* $u \in \mathrm{BMO}(D)$.

Combining this with Theorem 4 in Section 4 we have as a consequence

Corollary 2. *If D is any domain in $\mathbf{C}$ then* $\log \lambda_D \in \mathrm{BMO}(D)$.

F. W. Gehring has observed (unpublished) that the converse of Lemma 3 holds when u is harmonic.

Lemma 5. *If $u \in \mathrm{BMO}(D)$ is harmonic in D then* $\sup_{z\in D} d(z,\partial D)|\nabla u(z)| < \infty$.

These results allow us to prove

Theorem 6. *Let $D \subset \mathbf{C}$ and let $w = p(z)$ be an analytic covering map of B onto D. Then* $\log|p'| \in \mathrm{BMO}(B)$ *if and only if* $\inf_{w\in D} d(w,\partial D)\lambda_D(w) > 0$.

Proof. For the necessity suppose that $\log|p'| \in \mathrm{BMO}(B)$ and, by Lemma 5, that

$$\sup_{z\in B}(1-|z|)\left|\frac{p''(z)}{p'(z)}\right| = \sup_{z\in B} d(z,\partial B)|\nabla \log|p'(z)|| \leq b < \infty.$$

Then for any $z \in B$, $|p''(z)/p'(z)| \leq 2b\lambda_B(z)$ and we obtain from (12) and (13),

$$|\nabla \log \lambda_D(w)| \leq \frac{1}{|p'(z)|}\left(\left|\frac{p''(z)}{p'(z)}\right| + 2|z|\lambda_B(z)\right) < 2(b+1)\frac{\lambda_B(z)}{|p'(z)|} = 2(b+1)\lambda_D(w).$$

Therefore $d(w,\partial D)\lambda_D(w) \geq (1/2)(b+1)^{-1} > 0$ for all $w \in D$ by Theorem 5.

The proof of the sufficiency runs very much along the same lines using the bound in Theorem 4. If $\inf_{w\in D} d(w,\partial D)\lambda_D(w) \geq c > 0$ then

$$d(z,\partial B)\left|\frac{p''(z)}{p'(z)}\right| \leq d(z,\partial B)(|\nabla \log \lambda_D(w)|\,|p'(z)| + 2|z|\lambda_B(z)) \leq d(z,\partial B)\left(\frac{2|p'(z)|}{d(w,\partial B)} + 2|z|\lambda_B(z)\right)$$

$$\leq d(z,\partial B)\left(\frac{2}{c}\lambda_D(w)\,|p'(z)| + 2|z|\lambda_B(z)\right)$$
$$= \left(\frac{2}{c} + 2|z|\right) d(z,\partial B)\lambda_B(z)$$
$$< \left(\frac{2}{c} + 2\right) < \infty,$$

and we are done by Lemma 4.

When D is simply connected and hence p conformal, this theorem should be compared to a general theorem of H. M. Reimann [17] stating that the logarithm of the Jacobian of a quasiconformal mapping is of bounded mean oscillation.

Our final result in this area deals with the question of lifting functions of bounded mean oscillation from a domain to the disc. For convenience we shall say that a domain D has the BMO *lifting property* if for any $u \in \mathrm{BMO}(D)$ we have $u \circ p \in \mathrm{BMO}(B)$ when p is an analytic covering map of B onto D.

Theorem 7. *D has the* BMO *lifting property if and only if* $\inf_{z\in D} d(z,\partial D)\lambda_D(z) > 0$.

For the proof we require a special case of another theorem of H. M. Reimann [17], or rather a version of Reimann's theorem as given by P. Jones in [9].

Lemma 6. *If f is a conformal mapping of G onto D and $u \in \mathrm{BMO}(D)$ then $u \circ f \in \mathrm{BMO}(G)$ and $\|u \circ f\|_{*,G} \leq A\|u\|_{*,D}$ where A is an absolute constant.*

We should note that the general form of this theorem is for domains G, $D \subset \mathbf{R}^n$, $n \geq 2$, and f a K-quasiconformal mapping of G onto D. In this case the constant A depends upon n and K.

Proof of Theorem 7. We prove necessity first. From Corollary 2, $\log \lambda_D \in \mathrm{BMO}(D)$, $\log \lambda_B \in \mathrm{BMO}(B)$ and by hypothesis $\log \lambda_D \circ p \in \mathrm{BMO}(B)$ where p is an analytic covering map of B onto D. But then $\log |p'| = \log \lambda_B - \log \lambda_B \circ p$ is in $\mathrm{BMO}(B)$ and the desired conclusion follows from Theorem 6.

For the proof of sufficiency suppose that $\inf_{z\in D} d(z,\partial D)\lambda_D(z) \geq c > 0$ and let p be an analytic covering map of B onto D. By Lemma 3 there is a number $r = r(c) > 0$ such that p is univalent in each disc $\Delta \subset B$ of hyperbolic radius r. If Δ is such a disc then from (22)

$$\operatorname{diam} \Delta \leq kd(\Delta,\partial B)$$

where $k = 4s/(1 - s)^2$ and $s = \tanh r$. Next, if $u \in \mathrm{BMO}(D)$ then $u \circ p \in \mathrm{BMO}(\Delta)$ for each such Δ and

$$\|u \circ p\|_{*,\Delta} \leq A\|u\|_{*,p(\Delta)} \leq A\|u\|_{*,D}$$

by Lemma 6. Since Δ is a disc, we find in particular that

$$\frac{1}{m(\Delta)} \int_{\Delta} |u \circ p - (u \circ p)_{\Delta}| \, dm \leq A \|u\|_{*,D}$$

and according to the criterion cited at the beginning of this section, we have shown that $u \circ p \in \mathrm{BMO}(B)$ with $\|u \circ p\|_{*,B} \leq A'\|u\|_{*,D}$, $A' = A'(A,k)$.

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