THE QUASIHYPERBOLIC METRIC AND ASSOCIATED ESTIMATES ON THE HYPERBOLIC METRIC

By

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To Professor F. W. Gehring on his Sixtieth Birthday

§1. Introduction

In this paper we investigate the geometry of the quasihyperbolic metric of domains in $\mathbf{R}^n$. This metric arises from the conformally flat generalized Riemannian metric $d(x, \partial D)^{-1}|dx|$. Due to the fact that the density $d(x, \partial D)^{-1}$ is not necessarily differentiable, the classical theories of Riemannian geometry do not apply to this metric. The quasihyperbolic metric has been found to have many interesting and varied applications in geometric function theory. In particular, quasiconformal mappings are quasi-isometries of this metric for sufficiently far-lying points, also bounds on the quasihyperbolic metric in terms of other metrics imply that a domain is uniform which then implies certain injectivity criteria for locally-Lipschitz mappings, amongst others. In fact there is quite a strong relationship between uniform domains and the quasihyperbolic metric. Most of these basic results on the quasihyperbolic metric can be found in [3], [2] and [5]. We note here that the quasihyperbolic metric is complete and generates the usual topology on a proper subdomain of $\mathbf{R}^n$. Further, geodesics (length minimizing curves) always exist for this metric and these geodesics have Lipschitz continuous first derivatives, which is in fact best possible.

We begin by calculating the quasihyperbolic metric, its curvature and geodesics for some nontrivial examples in $\mathbf{R}^n$. We also calculate the possible isometries and show that they are conformal mappings, and so when $n > 2$ are Möbius transformations. We then turn to the planar case for a more detailed investigation. Here it becomes possible to compute the Gaussian curvature of the quasihyperbolic metric in some basic examples and use these for comparison in more general cases. In particular, we show that the quasihyperbolic metric of a planar domain is an S–K metric, in the sense of Heins, if and only if the domain is convex.

We denote euclidean n-space by $\mathbf{R}^n$ and its one point compactification by $\bar{\mathbf{R}}^n$.

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$B^n(x, r)$ is the euclidean ball centered at x with radius r, $\bar{B}^n(x, r)$ is its closure and $S^{n-1}(x, r)$ is its boundary. Throughout, D will denote a proper subdomain of $\mathbf{R}^n$ and $d(x, \partial D)$ the euclidean distance from $x \in D$ to the boundary of D. A Möbius transformation of $\bar{\mathbf{R}}^n$ is a finite composition of reflections in spheres and hyperplanes in $\mathbf{R}^n$.

We refer to [7] for the basic definitions and results concerning quasiconformal mappings. We note here, however, the important result that a 1-quasiconformal mapping of any subdomain of $\mathbf{R}^n$ is conformal and when $n > 2$ is a Möbius transformation. See [1].

§2. The quasihyperbolic metric

2.1. Definition. For any two points $x, y \in D$ we define the quasihyperbolic distance from x to y as

$$k_D(x, y) = \inf_\alpha \int_\alpha d(x, \partial D)^{-1} |dx|,$$

where the infimum is taken over all locally rectifiable arcs α joining x to y in D.

An arc for which the infimum is attained will be called a quasihyperbolic geodesic; as noted above such an arc always exists.

In the upper half-space $d(x, \partial D)^{-1}|dx| = x_n^{-1}|dx|$, which is the Riemannian metric associated to the hyperbolic metric of constant negative curvature equal to -1. Thus in the upper half-space the hyperbolic and quasihyperbolic metrics agree. This motivates the term "quasihyperbolic".

In the domain $D = \mathbf{R}^n \setminus \{0\}$ the quasihyperbolic metric is given by the formula

$$k_D(x, y) = \left[\left(\log \frac{|x|}{|y|} \right)^2 + \left(2 \sin^{-1} \left(\frac{1}{2} \left| \frac{x}{|x|} - \frac{y}{|y|} \right| \right) \right)^2 \right]^{1/2}.$$

To see this we first consider the case $n = 2$ where the exponential map $f(z) = e^z$ is a covering map of $\mathbf{C} = \mathbf{R}^2$ onto $D = \mathbf{R}^2 \setminus \{0\}$. We find easily that the quasihyperbolic metric $|z|^{-1}|dz|$ on D pulls back under f to the euclidean metric on $\mathbf{R}^2$. Thus the quasihyperbolic distance between two points $x, y \in D$ is the euclidean distance between the preimages of x and y which lie in the closure of a fundamental domain for $f(z) = e^z$. This is a horizontal strip of width 2π and a brief calculation gives the above formula. In the general case $n > 2$ the result follows by induction since each hyperplane through the origin is totally geodesic as it is the fixed point set of an isometry (the reflection through that hyperplane). We then see that the geodesics are logarithmic spirals lying in planes through the origin.

Since $\log |z|$ is harmonic in $\mathbf{R}^2 \setminus \{0\}$, the quasihyperbolic metric in $\mathbf{R}^2 \setminus \{0\}$ is flat (Gaussian curvature zero). In higher dimensions, for $D = \mathbf{R}^n \setminus \{0\}$, the sectional cuvature is zero for planes through the origin and positive for all others. In fact it is not difficult, although tedious, to show that the sectional curvature is

$$R_{ijij}(x) = 1 - \frac{x_i^2 + x_j^2}{|x|^2}.$$

In the previous two cases the quasihyperbolic density is smooth. The first natural domain to consider where this is not the case is the unit ball in $\mathbf{R}^n$, where a formula for the distance is surprisingly difficult to compute. The Riemannian metric tensor associated to the quasihyperbolic metric in the unit ball is

$$g_{ij}(x) = \delta_{ij}(1 - |x|)^{-2},$$

which is not differentiable at the origin. More generally, if $x \in D$ is a point such that there are at least two points y and z in the boundary of D for which $|x - y| = |x - z| = d(x, \partial D)$, then the quasihyperbolic density $d(x, \partial D)^{-1}$ is not differentiable at x. At such points geodesics are not usually locally unique and tend to bifurcate, see [5].

The curvature tensor for the quasihyperbolic metric in the ball is rather complicated; the sectional and Ricci curvatures are given respectively by $R_{ijij}(x) = -1/|x|(1 - |x|)^4$ and

$$R_{ij}(x) = -\delta_{ij}\left[\frac{n - 1 - (n-2)|x|}{|x|(1 - |x|)^2}\right] - (n-2)\left[\frac{\delta_{ij}(|x|^2 + |x|^3) - x_i x_j}{|x|^3(1 - |x|)^2}\right]$$

and it is not difficult to see that this is negative definite and radially symmetric. The Gaussian curvature in the unit disk is $-1/|x|$. These facts will later enable us to show that the only quasihyperbolic automorphisms of the unit ball are rotations and reflections through hyperplanes containing the origin.

We have seen that in general domains in $\mathbf{R}^n$ the sectional curvatures can be positive, zero or negative. We will show later that this is not the case for the quasihyperbolic metric in planar domains, which we will show always has nonpositive generalized Gaussian curvature.

We now seek the quasihyperbolic geodesics of the unit ball in $\mathbf{R}^n$. Unfortunately the solution involves rather complicated integrals and appears to have little geometric significance (as do the hyperbolic geodesics). However there is the advantage that the method generalizes sufficiently so that we may obtain a formula for the geodesics of any radially symmetric conformally flat Riemannian metric in the ball. Although the quasihyperbolic metric is not smooth at the origin, we can smooth it in any small neighbourhood and this will not change the geodesics that lie outside a slightly larger neighbourhood. In practice, however, we can ignore this detail since the quasihyperbolic geodesics through the origin are easily seen directly from the definition to be the radial line segments.

Thus let $p : [0, 1) \to (0, \infty)$ be a C^2 function defining a complete conformally flat

Riemannian metric on the ball by $p(|x|)|dx|$. Geodesics exist from classical theory and since rotations and reflections through hyperplanes containing the origin are isometries, we see that diameters are geodesics and that each plane through the origin is totally geodesic and so it suffices to consider the case $n = 2$.

Let α be a length minimizing geodesic which is not a diameter. By the local uniqueness and smoothness of geodesics we see that α intersects each diameter at most once. Hence we may assume by a rotation that α lies in the upper half-disk. We can then define a function $r(\theta)$ by

$$r(\theta) = |\alpha(t)|, \qquad \alpha(t) = [0, e^{i\theta}] \cap \alpha.$$

Thus there are angles θ^1 and θ^2 between which r is defined and we can reparametrize α as $r(\theta)e^{i\theta}$, for $\theta \in (\theta^1, \theta^2)$.

Geodesics will be the shortest curves with such a parametrization and so we seek the critical points of the variational problem to minimize over curves γ

$$\int_\gamma p(|x|)|dx|. \tag{2.2}$$

Since we know that geodesics have a parametrization as described above, we may vary over curves with such a parametrization. That is, we will vary over possible functions $r(\theta)$. This choice of parametrization reduces the nonlinear second order differential equation that we must solve to a first order separable equation considerably simplifying the problem.

In polar coordinates $|dx|^2 = dr^2 + (rd\theta)^2$. With our parametrization $dr = r'(\theta)d\theta$ and so the problem becomes to minimize over all functions $r(\theta)$

$$\int_{\theta^1}^{\theta^2} p(r(\theta))[r'(\theta)^2 + r(\theta)^2]^{1/2} d\theta. \tag{2.3}$$

The Euler–Lagrange equation for a critical solution to this problem is

$$\frac{\partial F}{\partial x}(\theta, r(\theta), r'(\theta)) - \frac{d}{d\theta}\left(\frac{\partial F}{\partial y}(\theta, r(\theta), r'(\theta))\right) = 0, \tag{2.4}$$

where $F(\theta, x, y) = p(x)[x^2 + y^2]^{1/2}$.

In the case that F, as above, is independent of θ, we see that since

$$\frac{d}{d\theta}\left(F - r'\frac{\partial f}{\partial y}\right) = r'\left(\frac{\partial F}{\partial x} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial y}\right)\right),$$

our critical solutions must satisfy

$$F(r, r') - r'\frac{\partial F}{\partial y}(r, r') = k,$$

where k is some constant determined by the initial value of r. Calculating the appropriate derivatives and simplifying we obtain

$$r(\theta)[p(r(\theta))^2 r(\theta)^2 - k^2]^{1/2} = kr'(\theta). \tag{2.5}$$

This equation is first order and separable. If we seek solutions that are perpendicular to the imaginary axis, there is no loss of generality in doing so, we see $r'(\frac{1}{2}\pi) = 0$ and so $k = \pm p(r(\frac{1}{2}\pi))r(\frac{1}{2}\pi)$.

For the quasihyperbolic metric $p(r) = (1-r)^{-1}$. The solution then is

$$\theta = k^2 \int \left[r^2 \left(\frac{r}{(1-r)^2} - k^2 \right) \right]^{-1/2} dr.$$

When $k = 1$ this integral is not too difficult to compute. We see in this case that $r(\frac{1}{2}\pi) = \frac{1}{2}$. Considering $\theta \geqq \frac{1}{2}\pi$ so that θ, as a function of r, is single valued, we find

$$\theta - \tfrac{1}{2}\pi = \sec^{-1}\left(\frac{r}{1-r}\right) - (2r-1)^{1/2}.$$

This is the equation of the quasihyperbolic geodesic through $\frac{1}{2}i$ perpendicular to the imaginary axis. When $r = 1$ we see $\theta = \pi - 1$, the endpoints of this geodesic are then e^i and $e^{i(\pi-1)}$. The endpoints of the hyperbolic geodesic through $\frac{1}{2}i$ and perpendicular to the imaginary axis are e^{it} and $e^{i(\pi-t)}$ where $t = \frac{1}{2}\pi - \tan^{-1}(3/4) \sim 0.9273$. One can then see that the quasihyperbolic geodesic lies above the hyperbolic geodesic. In general (2.5) will give an integral formula for the geodesics of any radially symmetric metric in the ball.

We now turn to consider the isometries of the quasihyperbolic metric of domains in $\mathbf{R}^n$. The following lemma shows that they are conformal and so when $n > 2$ are Möbius transformations.

2.6. Theorem. *Let p and p' be positive continuous functions on domains D and D' respectively. Let d and d' be the metrics associated with the conformally flat generalized Riemannian metrics $p(x)|dx|$ and $p'(x)|dx|$. If F: $(D, d) \to (D', d')$ is an isometry, then F is conformal.*

Proof. Since both metrics are clearly continuous with respect to the usual topology of $\mathbf{R}^n$ we see that F is a homeomorphism. We show that F is 1-quasiconformal, the result then follows as noted in the introduction. Let $x \in D$ and $\varepsilon > 0$. Then since p and p' are continuous and F is a homeomorphism there is a $\delta > 0$ such that if $|x - y| < \delta$, then

$$\begin{aligned} &1 - \varepsilon \leqq p(x)/p(y) \leqq 1 + \varepsilon, \\ &1 - \varepsilon \leqq p'(F(x))/p'(F(y)) \leqq 1 + \varepsilon. \end{aligned} \tag{2.7}$$

There is also a $\delta' > 0$ such that if $|x - y| < \delta'$ and if α and β are curves joining x to y and $F(x)$ to $F(y)$ respectively, with

$$d(x,y) \geqq (1-\varepsilon) \int_\alpha p(z)|dz|,$$

(2.8)

$$d'(F(x),F(y)) \geqq (1-\varepsilon) \int_\beta p(z)|dz|,$$

then $\alpha \subset B^n(x,\delta)$ and $\beta \subset F(B^n(x,\delta))$.

We then obtain from (2.7) and (2.8) the following estimates:

$$(1-\varepsilon)^2 |x-y| p(x) \leqq d(x,y) \leqq (1+\varepsilon)|x-y| p(x),$$

$$(1-\varepsilon)^2 |F(x)-F(y)| p'(F(x)) \leqq d'(F(x),F(y)) \leqq (1+\varepsilon)|F(x)-F(y)| p'(F(x)),$$

for all $|x-y| < \min\{\delta, \delta'\}$. Hence if $|x-y| = |x-z| = r < \min\{\delta,\delta'\}$, then

$$\begin{aligned} \frac{|F(x)-F(y)|}{|F(x)-F(z)|} &\leqq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 \frac{d'(F(x),F(y))}{d'(F(x),F(z))} \\ &= \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 \frac{d(x,y)}{d(x,z)} \\ &\leqq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^4. \end{aligned}$$

Now, since $\varepsilon > 0$ is arbitrary we see

$$\limsup_{r\to 0} \frac{\max\{|F(x)-F(y)| : |x-y| = r\}}{\min\{|F(x)-F(y)| : |x-y| = r\}} = 1.$$

Thus F is 1-quasiconformal according to the metric definition of quasiconformality and the proof is complete.

This is rather surprising in that even though the metric need not be differentiable the isometries are real analytic. We now see that since the isometries are smooth, they will preserve the curvature of the metric at points where the density is smooth. Thus in the case of the unit ball in $\mathbf{R}^n$ a quasihyperbolic automorphism (i.e., a quasihyperbolic self-isometry of B^n) must preserve the sectional curvature, which is radially symmetric and strictly decreasing in $|x|$. Thus such an automorphism is conformal and must map spheres of radius r to themselves. It is then easy to see that these maps must be either rotations or reflections in hyperplanes containing the origin. More generally we see that a quasihyperbolic isometry will preserve the singularities of the metric, that is, the points where the density is not differentiable. It is then easy to see that there are few quasihyperbolic isometries of a general domain in $\mathbf{R}^n$, and that most domains are only quasihyperbolically equivalent to images of themselves under euclidean isometries. This tends to suggest that, even in the plane, the quasihyperbolic metric is canonical and closely associated to the geometry of a given domain.

§3. The generalized Laplacian and Gaussian curvature

Henceforth D will be a proper subdomain of $\mathbf{C}$, the complex plane. If p is a positive C^2 function defining a conformal metric in D by $p(z)|dz|$, then the Gaussian curvature of this metric at $z_0 \in D$ is

$$\mathcal{K}_p(z_0) = -[\Delta \log p(z_0)]p(z_0)^{-2}. \tag{3.1}$$

This formula can be extended to define the Gaussian curvature of more general metrics if an integral formula for the Laplacian is used. If p is C^2 on D and if $\bar{B}(z_0, r) \subset D$ then an application of Green's theorem gives

$$\begin{aligned}\Delta p(z_0) &= \lim_{r \to 0} \frac{1}{\pi r^2} \int\limits_{B(z_0, r)} \Delta p \, dx \, dy \\ &= \lim_{r \to 0} \frac{1}{\pi r} \frac{d}{dr} \int\limits_0^{2\pi} p(z_0 + re^{i\theta}) d\theta\end{aligned}$$

which we write as the indeterminate form

$$\lim_{r \to 0} 4 \left(\frac{d}{dr} r^2\right)^{-1} \frac{d}{dr} \left[\frac{1}{2\pi} \int\limits_0^{2\pi} p(z_0 + re^{i\theta}) d\theta - p(z_0)\right].$$

Thus

$$\Delta p(z_0) = \lim_{r \to 0} \frac{4}{r^2} [m(p, r)(z_0) - p(z_0)] \tag{3.2}$$

where

$$m(p, r)(z_0) = \frac{1}{2\pi} \int\limits_0^{2\pi} p(z_0 + re^{i\theta}) d\theta$$

is the mean value function. It is also convenient to introduce

$$T(p, r)(z) = 4r^{-2}[m(p, r)(z) - p(z)]. \tag{3.3}$$

Formula (3.2) was first used by Heins in [4] for the purpose we propose. If p is semicontinuous (or even continuous) the limit in (3.2) need not exist. Supposing, though, that p is upper semicontinuous and positive, Heins defines the upper and lower Gaussian curvature of the conformal metric $p|dz|$ by

$$\bar{\mathcal{K}}_p(z) = -p(z)^{-2} \liminf_{r \to 0} T(\log p, r)(z),$$

$$\underline{\mathcal{K}}_p(z) = -p(z)^{-2} \limsup_{r \to 0} T(\log p, r)(z).$$

If $\bar{\mathcal{K}}_p(z) = \underline{\mathcal{K}}_p(z)$ then one may speak simply of the Gaussian curvature of $p\,|dz|$ at z, and we omit the bars on $\mathcal{K}_p$. We should point out that if $\log p$ is subharmonic, then $T(\log p, r)$ will converge to $\Delta \log p$ in the weak sense of measures and the Gaussian curvature defined as in (3.1) will exist as a distribution.

Heins calls a metric $p\,|dz|$ as above an S–K metric (subordinate in curvature) if $\bar{\mathcal{K}}_p(z) \leqq -1$. As an extension of the Ahlfors form of Schwarz's lemma, Heins shows that $p(z) \leqq \lambda_D(z)$, where $\lambda_D\,|dz|$ is the Poincaré metric of D with constant curvature -1, and if equality holds at one point it holds everywhere.

We now concentrate on the quasihyperbolic metric only and change notation to $\bar{\mathcal{K}}_D$ and $\underline{\mathcal{K}}_D$ to indicate the dependence on D. In connection with Heins' result we will show that the quasihyperbolic metric of a Jordan domain is an S–K metric if and only if the domain is convex. Our overall point, as illustrated by this particular case, is that the curvature can be computed for several examples and estimated in fairly general situations by means of the following comparison principles.

3.4. Lemma (Comparison Lemma). *If $z_0 \in D \subset D'$ and if $\zeta_0 \in \partial D \cap \partial D'$ with $|z_0 - \zeta_0| = d(z_0, \partial D) = d(z_0, \partial D')$ then $\bar{\mathcal{K}}_D(z_0) = \underline{\mathcal{K}}_{D'}(z_0)$.*

3.5. Lemma (Local comparison lemma). *If $z_0 \in D \cap D'$, $\zeta_0 \in \partial D \cap \partial D'$ with $|z_0 - \zeta_0| = d(z_0, \partial D) = d(z_0, \partial D')$ and if there is a neighborhood U of ζ_0 and a $\delta > 0$ such that $D \cap U \subset D' \cap U$ and $d(z_0, \partial D' \setminus U) \geqq d(z_0, \partial D') + \delta$ then $\bar{\mathcal{K}}_D(z_0) \leqq \underline{\mathcal{K}}_{D'}(z_0)$.*

In this and subsequent proofs we shall write

$$f_D(z) = \log d(z, \partial D)^{-1}$$

to simplify the notation.

Proof of Lemma 3.5. The hypotheses imply that if r is sufficiently small then all points on the circle $|z - z_0| = r$ satisfy $d(z, \partial D) \leqq d(z, \partial D')$. Thus for the mean values

$$m(f_D, r)(z_0) \geqq m(f_{D'}, r)(z_0)$$

and using $d(z_0, \partial D) = d(z_0, \partial D')$ twice we have first that

$$-T(f_D, r)(z_0) \leqq -T(f_{D'}, r)(z_0)$$

and second that

$$\bar{\mathcal{K}}_D(z_0) \leqq \underline{\mathcal{K}}_{D'}(z_0)$$

as desired.

The proof of Lemma 3.4 is the same.

3.6. Corollary. $\bar{\mathcal{K}}_D \leqq 0$ *for all* $D \subset \mathbf{C}$.

Proof. Let $z_0 \in D$ and $\zeta_0 \in \partial D$ with $|z_0 - \zeta_0| = d(z_0, \partial D)$. Let $D' = \mathbf{C} \setminus \{\zeta_0\}$. Then since $f_{D'}(z) = -\log|z - \zeta_0|$ is harmonic in D' we find from Lemma 3.4 that $\bar{\mathcal{K}}_D(z_0) \leqq \mathcal{K}_{D'}(z_0) = 0$.

This result is just the familiar fact that $\log d(z, \partial D)^{-1}$ is always subharmonic and is harmonic when D is the punctured plane. This is the only case when $\log d(z, \partial D)^{-1}$ is harmonic and later we shall prove the stronger result that $\bar{\mathcal{K}}_D = 0$ implies that D is the punctured plane. Thus if D has at least two boundary points, there are always points of strictly negative curvature.

3.7. Lemma. *If* $D = B(x, r)$, *then* $\mathcal{K}_D(z) = -r|x - z|^{-1} < -1$, $z \in D$, $\mathcal{K}_{\mathbf{C}\setminus\bar{D}}(z) = -r|x - z|^{-1} > -1$, $z \in \mathbf{C}\setminus\bar{D}$. *If D is a half-plane then* $\mathcal{K}_D = -1$.

Proof. The formulas for a disc and its complement are computations from (3.1). In a half-plane the quasihyperbolic and Poincaré metric are equal.

Note that the curvature is $-\infty$ at the center of a disc while it approaches zero at ∞ in the complement.

3.8. Proposition. *Let D be the exterior of a wedge of opening α, $0 \leqq \alpha < \pi$ and let L_1, L_2 be the rays perpendicular to ∂D at the vertex. Then $\mathcal{K}_D(z) = -1$ for points in either of the two quarter planes determined by L_1, L_2 and ∂D, $\mathcal{K}_D(z) = -\frac{1}{2}$ for points on L_1 or L_2 and $\mathcal{K}_D(z) = 0$ elsewhere in D.*

Proof. As the following argument will make clear, the opening α is irrelevant as long as $\alpha < \pi$, i.e., as long as D is not a half-plane. Thus we shall take D to be the plane slit from 0 to ∞ along the positive imaginary axis. Let $z_0 = x_0 + iy_0 \in D$. If $y_0 > 0$ then in any neighborhood of z_0 not meeting the real axis the quasihyperbolic metric of D is equal to the Poincaré metric of a half-plane and so has curvature -1. Similarly if $y_0 < 0$ then in a small neighborhood of z_0 the quasihyperbolic metric of D is that of the punctured plane and so has curvature zero. The case we must compute is when $y_0 = 0$ and we may take $z_0 = x_0 > 0$. Let $r < x_0$ be small. If $z = x_0 + re^{i\theta}$ then

$$d(z, \partial D) = \begin{cases} x_0 + r\cos\theta, & 0 \leqq \theta \leqq \pi \\ (x_0^2 + r^2 + 2x_0 r(\cos\theta))^{1/2}, & \pi \leqq \theta \leqq 2\pi. \end{cases}$$

Hence for $0 \leqq \theta \leqq \pi$ we find, using a Taylor expansion,

$$\begin{aligned} \frac{1}{2\pi}\int_0^\pi \log d(z, \partial D)^{-1} d\theta - \tfrac{1}{2}\log d(z_0, \partial D)^{-1} &= -\frac{1}{2\pi}\int_0^\pi \log\left(1 + \frac{r}{x_0}\cos\theta\right) d\theta \\ &= \frac{r^2}{8x_0^2} + O(r^3), \end{aligned} \tag{3.9}$$

while for $\pi \leqq \theta \leqq 2\pi$

$$\frac{1}{2\pi}\int_{\pi}^{2\pi} \log d(z,\partial D)^{-1}d\theta - \tfrac{1}{2}\log d(z_0,\partial D)^{-1} = -\frac{1}{4\pi}\int_{\pi}^{2\pi} \log\left(1+\frac{r^2}{x_0^2}+\frac{2r}{x_0}\cos\theta\right)d\theta$$

$$= -\frac{1}{4\pi}\left[\frac{\pi r^2}{x_0^2}-\frac{2r^2}{x_0^2}\int_{\pi}^{2\pi}\cos^2\theta\, d\theta + O(r^3)\right]$$

$$= O(r^3).$$

Adding these and recalling (3.3) gives

$$T(f_D, r)(z_0) = \frac{1}{2x_0^2} + o(1)$$

and so the curvature is $-\frac{1}{2}$ as claimed.

We continue with one more example.

3.10. Proposition. *Let $D = \mathbf{C}\setminus\{z_1, z_2\}$ and let L be the perpendicular bisector of the line segment $[z_1, z_2]$. Then*

$$\mathcal{K}_D(z) = \begin{cases} 0, & z \notin L, \\ -\infty, & z \in L. \end{cases}$$

Proof. The case when $z \notin L$ is clear by comparison to $\mathbf{C}\setminus\{z_1\}$ or $\mathbf{C}\setminus\{z_2\}$. We may assume that L is the real axis and that $z_1 = ti$, $z_2 = -ti$, $t > 0$. Let $x \in L$, without loss of generality we can take $x \leqq 0$, let $d = |x - it| = d(x, \partial D)$ and let $r < d$, r small. To estimate the mean value of $f_D(z) = \log d(z, \partial D)^{-1}$ on the circle $|z - x| = r$ it will be enough, by symmetry, to consider the integral over the top half. For $0 \leqq \theta \leqq \pi$ we find that

$$|x + re^{i\theta} - ti|^2 = d^2 + r^2 - 2rd\cos(\theta - \psi)$$

where $\sin\psi = t/d > 0$. Then

$$\frac{1}{2\pi}\int_0^{\pi} \log d(z,\partial D)^{-1}d\theta - \tfrac{1}{2}\log d(x,\partial D)^{-1}$$

$$= -\frac{1}{4\pi}\int_0^{\pi}\log\left(1+\frac{r^2}{d^2}-\frac{2r}{d}\cos(\theta-\psi)\right)d\theta$$

$$= -\frac{1}{4}\frac{r^2}{d^2}+\frac{1}{2\pi}\int_0^{\pi}\left(\frac{r}{d}\cos(\theta-\psi)+\frac{r^2}{d^2}\cos^2(\theta-\psi)\right)d\theta + O(r^3)$$

$$= \frac{rt}{\pi d^2} + O(r^3).$$

Adding the integral over the bottom half of the circle and multiplying by $4r^{-2}$ gives

$$T(f_D, r)(x) = \frac{8t}{\pi d^2}\frac{1}{r} + o(1),$$

which tends to $+\infty$ as $r \to 0$ implying $\mathcal{K}_D(x) = -\infty$.

We get a better perspective on the preceding proposition and a sharper focus on subsequent remarks if we make the following definition.

3.11. Definition. We say a point $z \in D$ is *centered in* D if the circle $|w - z| = d(z, \partial D)$ meets ∂D in at least two points. If the circle meets ∂D in one point we say that z is *uncentered.*

Thus the points of the line in the preceding proposition are the centered points in D, as are the points on the line bisecting the vertex angle α, $0 < \alpha < \pi$, in the interior of a wedge of opening α. In general

3.12. Corollary. *If $z \in D$ is a centered point then $\mathcal{K}_D(z) = -\infty$.*

Proof. Let z_1 and z_2 be points of the boundary of D for which $|z - z_1| = |z - z_2| = d(z, \partial D)$. Apply the comparison lemma 3.4 and the preceding proposition to D and $\mathbf{C} \setminus \{z_1, z_2\}$.

It is possible to have an uncentered point where $\mathcal{K}_D(z) = -\infty$. Let D be the interior of the parabola $y = x^2$ and let $z_0 = \frac{1}{2}i$. Then z_0 is an uncentered point in D, the origin being the unique closest boundary point, but $\mathcal{K}_D(z_0) = -\infty$ as we shall now show. Let $\varepsilon > 0$ be small and let D' be the disc $|z - (\frac{1}{2} + \varepsilon)i| = \frac{1}{2} + \varepsilon$. Then $\partial D'$ meets ∂D at two points where $y = 2\varepsilon$ but $d(z_0, \partial D') = d(z_0, \partial D)$. The local comparison lemma 3.5 applies in this case to D and D' and from this and Lemma 3.7, $\bar{\mathcal{K}}_D(z_0) \leqq \underline{\mathcal{K}}_{D'}(z_0) = -(\frac{1}{2} + \varepsilon)\varepsilon^{-1}$. Let $\varepsilon \to 0$.

We can now return to a point raised earlier.

3.13. Theorem. *If $\bar{\mathcal{K}}_D = 0$ then D is the punctured plane.*

For the proof we require first a rather off-beat characterization of convex sets via uncentered points.

3.14. Lemma. *D contains only uncentered points if and only if $\mathbf{C} \setminus D$ is convex.*

Perhaps surprisingly, this does not seem to be known. We only need necessity in our later work, but sufficiency is easy enough. The proof uses a suggestion of D. Stowe.

Proof. We prove necessity first. We define a map $P\colon D \to D$ as follows. Let $z \in D$ and draw the segment from z to its closest point $\zeta \in \partial D$. There is only one such point by hypothesis. Define $P(z)$ to be the point on this segment with $d(P(z), \partial D) = \frac{2}{3}d(z, \partial D)$, i.e. move z one third of the way to the boundary. Note that all points on the segment from z to ζ have ζ as their closest boundary point and that points on the segment then move toward ζ under repeated applications of P. It follows that P is one-to-one. It is also a simple consequence of the hypothesis that P is continuous and then has a continuous extension to $\bar{D}$ by defining it to be the identity on ∂D.

Fix $z_0 \in D$ and let $\bar{B}_0 = \bar{B}(z_0, d(z_0, \partial D))$. Then $\bar{B}_0$ meets ∂D at exactly one point ζ_0. Let R be the open ray from ζ_0 through z_0. We claim that $R \subset D$ and furthermore that if z is any point on R then $d(z, \partial D) = |z - \zeta_0|$. To prove this we show first that there is a point $z_1 \in B_0$ with $P(z_1) = z_0$, observe that such a z_1 must lie on R. The simplest argument is topological. If $\gamma = \partial B_0$ and $\Gamma = P(\gamma)$ then Γ is a Jordan curve and $z_0 \notin \Gamma$ for no point of γ can be moved as far as to z_0 under P. Next, there is an obvious homotopy of maps between id $| \partial B_0$ and $P | \partial B_0$ so that the winding numbers are constant, $1 = n(\gamma, z_0) = n(\Gamma, z_0)$. Finally, a moment's thought shows that $P | B_0$ is a homeomorphism onto its range and thus there must be a $z_1 \in B_0$ with $P(z_1) = z_0$ as desired. As mentioned above $z_1 \in R$ and hence $d(z_1, \partial D) = |z_1 - \zeta_0| = \frac{3}{2} d(z_1, \partial D)$. We have thus established that all points along R from ζ_0 to z_1 have ζ_0 as their closest point on ∂D and they lie in D because they lie in B_0. Now repeat this entire argument with z_0 replaced by z_1 and continue in this fashion. The claim is proved.

By way of contradiction, suppose next that $\mathbf{C} \setminus D$ is not convex. Then there is a segment $\beta \subset D$ with endpoints $z_1, z_2 \in \partial D$. Take β to be an interval on the real axis and let α be the part of ∂D from z_1 to z_2, lying below β. (If all of ∂D is below β then α is the part nearest β in the obvious sense.) Let z_0 be the midpoint of β and let ζ_0 be the closest point on ∂D to z_0. Since $z_0 \in D$ is uncentered $\zeta_0 \neq z_1, z_2$ and since the open ray R from ζ_0 through z_0 lies in D it follows that $\zeta_0 \in \alpha \setminus \{z_1, z_2\}$. But now if z is far enough out on R then z will be closer to one (or both) of z_1, z_2 than it is to ζ_0 and this contradicts the second property of points on R. Hence $\mathbf{C} \setminus D$ is convex and the necessity is proved.

The proof of sufficiency is very simple. If $z_0 \in D$ is a centered point then $\bar{B}(z_0, d(z_0, \partial D))$ meets ∂D in (at least) two points, say ζ_1, ζ_2. The chord between ζ_1 and ζ_2 lies in D showing that $\mathbf{C} \setminus D$ is not convex.

Returning to general considerations, we say that a point $\zeta \in \partial D$ is *unmasked* with respect to D if it is the closest boundary point to some $z \in D$.

We shall need the following simple consequence of the preceding Lemma.

3.15. Corollary. *If all points of D are uncentered then all points of ∂D are unmasked.*

Proof. Since $\mathbf{C} \setminus D$ is convex, for any $\zeta \in \partial D$ we can find a supporting line L at ζ for $\mathbf{C} \setminus D$ and a point $z \in D$ so that $d(z, \partial D) = d(z, L) = |z - \zeta|$.

Proof of Theorem 3.13. By Corollary 3.12, $\mathcal{K}_D(z) = -\infty$ at any centered point. Consequently D contains only uncentered points and by Lemma 3.14 $\mathbf{C}\setminus D$ is convex. D cannot be a half-plane by Lemma 3.7.

First suppose that $D^* = \mathbf{C}\setminus D$ contains an open set. Let $z_0 \in \operatorname{Int} D^*$ and let $r = d(z_0, \partial D^*) = d(z_0, \partial D)$. Then $B(z_0, r) \subset D^*$ and $\partial B(z_0, r)$ meets ∂D^* in a point ζ which is an unmasked point for ∂D by Corollary 3.15. Let $z \in D$ be such that $|z - \zeta| = d(z, \partial D)$ and let $D' = \mathbf{C}\setminus \bar{B}(z_0, r)$. Then by the comparison lemma 3.4 and Lemma 3.7, $\bar{\mathcal{K}}_D(z) \leqq \underline{\mathcal{K}}_{D'}(z) = -r|z - z_0|^{-1} < 0$ contradicting the hypothesis.

Thus D is dense in $\mathbf{C}$ and ∂D must be a proper connected subset of a straight line. Let $\zeta \in \partial D$. If ζ is an endpoint of an interval then, as in the proof of Proposition 3.8, we can find $z \in D$ with $d(z, \partial D) = |z - \zeta|$ and $\mathcal{K}_D(z) = -\frac{1}{2}$. If ζ is an interior point of an interval then we can likewise find $z \in D$ with $\mathcal{K}_D(z) = -1$. The only possibility consistent with the hypothesis is that ∂D reduces to a single point as was to be shown.

Nonzero, finite upper and lower bounds for the curvature can be obtained by the comparison lemmas in the presence of additional geometric information on the boundary.

3.16. Definition. We say that a point $z \in D$ is a *point of convexity* if z is an uncentered point and if the following condition is satisfied. Let $\zeta \in \partial D$ be the closest point to z, let L be the line through ζ tangent to $B(z, d(z, \partial D))$ and let H^+ be the component of $\mathbf{C}\setminus L$ containing z. Then there exists a neighborhood U of ζ such that ∂D is separating in U and $U \cap \partial D \subset \bar{H}^+$. We say an uncentered point is a *point of concavity* if the above condition is satisfied for some neighborhood U with H^+ replaced by H^-, the component of $\mathbf{C}\setminus L$ not containing z. A point is a point of strict convexity or concavity if we may choose L to be a circle.

Note that if D is convex then each uncentered point is a point of convexity while each uncentered point of $\mathbf{C}\setminus D$ is a point of concavity.

3.17. Theorem. *Let $z \in D$. Then*

(i) *If z is a point of convexity, $\bar{\mathcal{K}}_D(z) \leqq -1$ and the inequality is strict if z is a point of strict convexity.*

(ii) *If z is a point of concavity, $\underline{\mathcal{K}}_D(z) \geqq -1$ and the inequality is strict if z is a point of strict concavity.*

Proof. The fact that in each case z is an uncentered point in both D and in the respective upper and lower half-planes allows us to apply the local comparison lemma 3.5. The desired inequality, including the case of strict inequality, follow from this and from Lemma 3.7.

3.18. Corollary. *If D is convex, then $\bar{\mathcal{K}}_D \leqq -1$ and hence the quasihyperbolic metric is an S–K metric.*

3.19. Corollary. *If* $\inf \underline{\mathcal{K}}_D \neq -\infty$ *then* $\underline{\mathcal{K}}_D \geqq -1$.

For as before every point of D is uncentered, $\mathbf{C}\setminus D$ is convex and each point of D is then a point of concavity as mentioned above.

From the result of Heins on S–K metrics

3.20. Corollary. *If D is convex, then $d(z,\partial D)^{-1} \leqq \lambda_D(z)$ for all $z \in D$.*

Next, we can show

3.21. Proposition. *The quasihyperbolic metric of a Jordan domain is an S–K metric if and only if D is convex.*

This follows directly from the preceding results and

3.22. Lemma. *If D is a Jordan domain which is not convex, then D contains a point of strict concavity.*

Proof of Lemma 3.22. Let $\hat{D}$ be the convex hull of D. If $\beta = \partial\hat{D}$ then β is a convex Jordan curve and $\beta\setminus\partial D$ is a nonempty collection of disjoint open line segments. We may suppose by rotation and translation that $\hat{D} \subset H^-$, the lower half-plane, and that the line segment $(-a,a)$ is a component of $\beta\setminus\partial D$. Let α be the subarc of ∂D joining $-a$ to a and such that the domain bounded by $[-a,a]\cup\alpha$ contains no points of D. If $r = d(0,\alpha)$, then $0 < r \leqq a$. Let $t = (2r)^{-1}(a^2 - r^2)$. Then by construction $B(ti, |ti - a|)$ cannot contain α. Thus let s be the smallest number such that $\alpha \subset \bar{B}(ti,s) = \bar{B}$. Then $-a$ and a are interior points of B. Let $w \in \alpha \cap \partial B$ and let $c = d(w,\partial D\setminus\alpha) > 0$. It is clear that $B(w,c)\setminus B \subset D$, for the only boundary points of D in $B(z,c)$ are those of α which lie in B. Next let $z = w + \frac{1}{4}cw|w|^{-1}$. Then z is the point of strict concavity we seek, for the closest point of the boundary of D to z is w and the concavity condition is easily seen to be satisfied with $U = B(w,\frac{1}{4}c)$ and $L = \partial B$.

To conclude this aspect of the discussion and as a complement to Theorems 3.13 and 3.17 we have

3.23. Theorem. *If $\mathcal{K}_D$ is constant and negative then D is a half-plane and $\mathcal{K}_D = -1$.*

Proof. Once again, D can contain only uncentered points and $\mathbf{C}\setminus D$ is convex. Observe first that D cannot be dense in $\mathbf{C}$. For if so then ∂D must be a proper connected subset of a straight line and if ζ is an endpoint of ∂D then we can find a point $z \in D$ with $d(z,\partial D) = |z-\zeta|$ for which $\mathcal{K}_D(z) = 0$ as in Proposition 3.8. Next, if ∂D contains a line segment then Proposition 3.8 again implies that $\mathcal{K}_D = -1$ for points in D near this segment and hence, by hypothesis, that $\mathcal{K}_D = -1$ in all of D.

Now let $\zeta \in \partial D$ and choose a $z_0 \in D$ and a supporting line L for $\mathbf{C}\setminus D$ at ζ such that $d(z_0,\partial D) = d(z_0,L) = |z_0 - \zeta|$. Let R be open ray from ζ through z_0. Then $R \subset D$, $R \perp L$ and $d(z,\partial D) = |z-\zeta|$ for all $z \in R$. We consider circles C tangent to L at ζ whose centers lie in the half-plane containing $\mathbf{C}\setminus D$. For these circles

$$d(z, C) = d(z, \partial D) = |z - \zeta| \qquad \text{for all } z \in R.$$

Let U be a small neighborhood of ζ, write $\alpha = \partial D \cap U$, $\beta = L \cap U$ and let α_1, α_2 and β_1, β_2 denote the components of $\alpha\setminus\{\zeta\}$ and $\beta\setminus\{\zeta\}$ respectively. We distinguish four cases; only the first two do not lead to a contradiction.

(i) $\alpha = \beta$ for some U.

(ii) $\alpha \neq \beta$ but for every U no circle C separates $\alpha\setminus\{\zeta\}$ from $\beta\setminus\{\zeta\}$; thus α is squeezed between β and every circle C tangent to β at ζ. It follows easily that α is smooth with curvature zero at ζ.

(iii) $\alpha_1 = \beta_1$, say, and there is a circle C separating α_2 from β_2. In this case ∂D contains a line segment, β_1, and so $\mathcal{K}_D \equiv -1$. But if C has radius r and center w we obtain a contradiction from the estimate $-1 = \mathcal{K}_D(z) \geqq -\frac{1}{2} - \frac{1}{2}r|z - w|^{-1} > -1$ for all $z \in R$. This is a calculation, much as we have done before, using eq. (3.9) in the proof of Proposition 3.8 and (literally half of) the curvature calculation for the exterior of a circle (via the integral formula for the curvature rather than differentiating to find the Laplacian). We omit the details.

(iv) There is a circle C separating $\alpha\setminus\{\zeta\}$ from $\beta\setminus\{\zeta\}$. Again if the circle has radius r and center w then the local comparison lemma 3.5 and Lemma 3.7 give $\mathcal{K}_D(z) \geqq -r|z - w|^{-1}$ for all $z \in R$. But $\mathcal{K}_D$ is a negative constant while $-r|z - w|^{-1} \to 0$ as $z \to \infty$ along R and a contradiction ensures.

We conclude that ∂D has curvature zero everywhere, hence ∂D is a straight line, D is a half-plane and $\mathcal{K}_D = -1$.

3.24. Remark. Corollary 3.20 giving the lower bound $\lambda_D(z) \geqq d(z, \partial D)^{-1}$ for the Poincaré metric of a convex domain is well known and follows easily from Schwarz's lemma. With an eye toward possible generalizations to higher dimensions and to surfaces, we wish to point out that an advantage of the approach here, aside from the more general Theorem 3.17, is the independence from conformal mapping and the simple geometric estimates. (Recall that the Poincaré metric can be defined as the maximal solution of $\Delta \log u = u^2$. Generalizations of this to $\mathbf{R}^n$ are being pursued by A. Weitsman.) For example, Theorem 3.23, which contrasts sharply the differential geometric properties of the hyperbolic and quasihyperbolic metric, *can* be deduced from Heins' work and a careful examination of covering maps (see [6]). So in a sense, here we have replaced the uniformization theorem by far more elementary considerations. We hope that these ideas will be of further use.

On the subject of upper bounds for the curvature, if $\bar{\mathcal{K}}_D(z) \leqq -a^2$, $a > 0$ then the metric defined by $ad(z, \partial D)^{-1}|dz|$ will have upper Gaussian curvature $\leqq -1$ and this gives the estimate $ad(z, \partial D)^{-1} \leqq \lambda_D(z)$. To take the simplest example let $A = A(1, R)$ be the annulus of radii 1 and $R > 1$. Then from our previous calculations we see that

$$\bar{\mathcal{K}}_A(z) = \begin{cases} -|z|^{-1}, & |z| < (1+R)/2 \\ -\infty, & |z| = (1+R)/2 \\ -R|z|^{-1}, & |z| > (1+R)/2, \end{cases}$$

and so $\bar{\mathcal{K}}_A(z) \leqq -2(1+R)^{-1}$ which gives $(2(1+R)^{-1})^{1/2} d(z, \partial A)^{-1} \leqq \lambda_A(z)$.

Finally, we give two results on lower bounds for the curvature at the boundary.

3.25. Proposition. *If D is any subdomain of $\mathbf{C}$ then* $\liminf_{z \to \zeta} \underline{\mathcal{K}}_D(\zeta) \geqq -1$ *where $\zeta \in \partial D$ is any unmasked point and z approaches ζ in any disc contained in D meeting ∂D at ζ.*

Proof. Let ζ be as described and let $z_0 \in D$ be such that $d = d(z_0, \partial D) = |z_0 - \zeta|$. If z is any point in the disc $B = B(z_0, d)$ then by the comparison lemma 3.4, $\underline{\mathcal{K}}_D(z) \geqq \bar{\mathcal{K}}_B(z) = -d|z - z_0|^{-1}$ which tends to -1 as $z \to \zeta$ in B.

Thus for a convex domain $\mathcal{K}_D(\zeta) = -1$ at all unmasked boundary points, with the boundary values understood as above.

3.26. Proposition. *Suppose there are positive constants b and c such that if $d(z, \partial D) < b$ then there is a disc of radius c, B_c, such that $B(z, d(s, \partial D)) \subset B_c \subset D$. Then*

$$\underline{\mathcal{K}}_D(z) \geqq -\frac{c}{c - d(z, \partial D)}$$

for all $z \in D$ with $d(z, \partial D) < \min\{b, c\}$.

The proof is similar to that of the preceding proposition. The condition says that the boundary of D is uniformly osculated by balls of radius c. This condition is satisfied, for example, if ∂D is C^2 and compact since we can find uniform bounds on the curvature of ∂D to obtain c and the length of an inward directed normal to ∂D to find b. So if D is a bounded domain with a C^2 boundary and ζ is any boundary point then $\liminf_{n \to \infty} \underline{\mathcal{K}}_D(z_n) \geqq -1$ for any sequence of points $z_n \in D$ converging to ζ. Note that in a square, if $\{z_n\}$ is a sequence of points approaching a corner, which is a masked boundary point, then $\mathcal{K}(z_n)$ is $-\infty$ if z_n is on a diagonal and -1 if it is off the diagonal. The lim inf is $-\infty$.

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