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Curvature Properties of Planar Harmonic Mappings

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Dedicated to the memory of Walter Hengartner

Abstract. A geometric interpretation of the Schwarzian of a harmonic mapping is given in terms of geodesic curvature on the associated minimal surface, generalizing a classical formula for analytic functions. A formula for curvature of image arcs under harmonic mappings is applied to derive a known result on concavity of the boundary. It is also used to characterize fully convex mappings, which are related to fully starlike mappings through a harmonic analogue of Alexander's theorem.

Keywords. Harmonic mapping, Schwarzian derivative, geodesic curvature, minimal surface, convex, starlike.

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1. Introduction

A planar harmonic mapping is a complex-valued harmonic function

$$f(z) = u(z) + iv(z), \qquad z = x + iy,$$

defined on some domain $D \subset \mathbb{C}$. If D is simply connected, the mapping has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in D and $g(z_0) = 0$ for some specified point $z_0 \in D$. The mapping f is locally univalent if and only if its Jacobian $|h'|^2 - |g'|^2$ does not vanish. It is said to be orientation-preserving if $|h'(z)| > |g'(z)|$ in D, or equivalently if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property $|\omega(z)| < 1$ in D.

In two previous papers [3, 4] we introduced a notion of Schwarzian derivative for a locally univalent harmonic mapping and showed that it retains some of the classical properties of the Schwarzian of an analytic function. In these investigations it was fruitful to identify the harmonic mapping with its local lift to a minimal surface. One purpose of the present paper is to develop a geometric interpretation of the Schwarzian in terms of change of geodesic curvature, thus directly generalizing a known property of Schwarzians for analytic functions. In

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fact, this property is known to hold with respect to a more general notion of Schwarzian for mappings between Riemannian manifolds, according to work of Osgood and Stowe [9]. However, we have considered it worthwhile to derive the result independently for harmonic mappings, appealing only to basic concepts of differential geometry in $\mathbb{R}^3$.

According to the Weierstrass–Enneper formulas, an orientation-preserving harmonic mapping $f = h + \overline{g}$ with dilatation $\omega = q^2$, the square of an analytic function q, lifts locally to a minimal surface expressed by conformal parameters. The Cartesian coordinates $\langle u, v, w\rangle$ of the surface are given by

$$u(z) = \operatorname{Re}\{f(z)\},$$
$$v(z) = \operatorname{Im}\{f(z)\},$$
$$w(z) = 2\operatorname{Im}\left\{\int_{z_0}^{z} h'(\zeta)q(\zeta)\,d\zeta\right\}.$$

The first fundamental form of the surface is $ds^2 = \lambda^2|dz|^2$, where the conformal metric $\lambda = e^\sigma$ has the form $\lambda = |h'| + |g'|$. Further information about harmonic mappings and their relation to minimal surfaces can be found in the book [5].

We begin this paper with a formula for the geodesic curvature of a curve on the minimal surface that is lifted from a curve in the range of the harmonic mapping f. The derivative of geodesic curvature with respect to arclength is then expressed in terms of the Schwarzian derivative of f.

The remainder of the paper concerns univalent harmonic mappings in the unit disk, normalized so that $f(0) = 0$. A formula is derived for the curvature of the local image of a circle $|z| = r$. When f has a smooth extension to an arc of the unit circle where its dilatation has unit modulus, the formula shows that the image arc is *concave*. This is a known result [8, 6, 1], but the natural proof based on curvature appears to be new.

When the dilatation is a finite Blaschke product, the curvature formula leads to a kind of Gauss–Bonnet formula that relates total curvature of the boundary with angles at corners. This is then applied to interpret certain geometric features of harmonic shears of analytic functions convex in one direction, when the dilatation imposed is a finite Blaschke product.

The curvature formula also allows us to give a simple criterion for a harmonic mapping to be fully convex; that is, to map each circle $|z| = r < 1$ onto a convex curve. It is well known that the hereditary property of convex analytic mappings does not generalize to harmonic mappings. More precisely, a harmonic mapping may send the disk univalently onto a convex domain, yet the image of the disk $|z| = r$ may fail to be convex when $\sqrt{2} - 1 < r < 1$. The lower bound $\sqrt{2} - 1$ is sharp, as Ruscheweyh and Salinas [10] proved. A criterion is also developed for a harmonic mapping to be fully starlike, and the two criteria are applied to give a harmonic version of Alexander's theorem that an analytic function f is convex if and only if $zf'(z)$ is starlike. However, a harmonic mapping may be fully starlike and yet fail to be univalent.

2. Geodesic curvature on the minimal surface

Suppose that

$$f(z) = h(z) + \overline{g(z)} = u(z) + iv(z)$$

is a locally univalent orientation-preserving harmonic mapping with dilatation $g'/h' = q^2$, so that it lifts locally to a minimal surface Ω of height $w = 2\,\mathrm{Im}\{\Phi(z)\}$, where

$$\Phi(z) = \int_{z_0}^{z} h'(\zeta)q(\zeta)\,d\zeta.$$

Let $z = z(t)$ define a curve C in the plane, parametrized by its arclength t, so that $z'(t) = e^{i\theta(t)}$ is the unit tangent vector and $\kappa(t) = \theta'(t)$ is the curvature of C. Note that $z'' = i\kappa z'$. With slight abuse of notation, let

$$\langle u, v, w\rangle = \varphi(t) = \langle h(z(t)) + \overline{g(z(t))}, 2\,\mathrm{Im}\{\Phi(z(t))\}\rangle$$

denote the lift of C to a curve Γ on Ω. Then a tangent vector to Γ is

$$\varphi'(t) = \langle h'(z(t))z'(t) + \overline{g'(z(t))z'(t)}, 2\,\mathrm{Im}\{h'((z(t))q(z(t))z'(t)\}\rangle, \tag{1}$$

and the element of arclength of Γ is $ds = e^{\sigma}\,dt$, where $e^{\sigma} = |h'| + |g'|$. To define a canonical normal vector to φ' in the tangent plane of Ω, we can replace z' by iz' in the formula for φ', since we are using conformal coordinates. Thus we obtain the normal vector

$$\mathbf{n} = \langle ih'z' - i\,\overline{g'z'}, 2\,\mathrm{Re}\{h'qz'\}\rangle, \tag{2}$$

and the unit normal vector is $\mathbf{N} = e^{-\sigma}\mathbf{n}$. Similarly, the unit tangent vector to the curve Γ is $\mathbf{T} = e^{-\sigma}\varphi'$, and the curvature vector is $d\mathbf{T}/ds$. The projection of the curvature vector onto the tangent plane of Ω is the same as its projection onto $\mathbf{N}$, since $\mathbf{T}\cdot d\mathbf{T}/ds = 0$. Thus the geodesic curvature of Γ with respect to Ω has the expression

$$\begin{aligned}\widehat{\kappa} &= \frac{d\mathbf{T}}{ds}\cdot\mathbf{N} = e^{-\sigma}\frac{d\mathbf{T}}{dt}\cdot\big(e^{-\sigma}\mathbf{n}\big)\\ &= e^{-2\sigma}\frac{d}{dt}\big(e^{-\sigma}\varphi'(t)\big)\cdot\mathbf{n} = e^{-3\sigma}\varphi''(t)\cdot\mathbf{n},\end{aligned} \tag{3}$$

since $\varphi'(t)\cdot\mathbf{n} = 0$.

We will show that the geodesic curvature $\widehat{\kappa}$ is expressed by the formula

$$e^{\sigma}\widehat{\kappa} = \kappa - \frac{\partial\sigma}{\partial(iz')}, \tag{4}$$

where $\partial/\partial(iz')$ indicates a derivative in the direction of the vector iz' normal to the curve C. Since iz' is a unit vector and

$$2\frac{\partial\sigma}{\partial\overline{z}} = \frac{\partial\sigma}{\partial x} + i\frac{\partial\sigma}{\partial y}$$

is the complex form of the gradient of σ, the relation (4) can be recast as

$$e^{\sigma}\widehat{\kappa} = \kappa - 2\,\mathrm{Re}\left\{iz'\,\overline{\frac{\partial\sigma}{\partial\overline{z}}}\right\} = \kappa + 2\,\mathrm{Im}\left\{\frac{\partial\sigma}{\partial z}z'\right\}. \tag{5}$$

For further reduction of the formula, we need to calculate the derivative $\partial\sigma/\partial z$. Since $\sigma = \log(|h'| + |g'|)$, we find that

$$\frac{\partial\sigma}{\partial z} = e^{-\sigma}\left(\frac{\partial|h'|}{\partial z} + \frac{\partial|g'|}{\partial z}\right) = \tfrac{1}{2}e^{-\sigma}\left(\frac{h''}{h'}|h'| + \frac{g''}{g'}|g'|\right), \tag{6}$$

so that (5) becomes

$$e^{\sigma}\widehat{\kappa} = \kappa + e^{-\sigma}\,\mathrm{Im}\left\{\left(\frac{h''}{h'}|h'| + \frac{g''}{g'}|g'|\right)z'\right\}. \tag{7}$$

But $g' = q^2 h'$ and therefore

$$\frac{g''}{g'} = \frac{h''}{h'} + 2\frac{q'}{q}, \tag{8}$$

so an equivalent form is

$$e^{\sigma}\widehat{\kappa} = \kappa + \mathrm{Im}\left\{\frac{h''}{h'}z'\right\} + \frac{2}{1+|q|^2}\,\mathrm{Im}\{\overline{q}q'z'\}, \tag{9}$$

because

$$e^{-\sigma}|h'| = \frac{1}{1+|q|^2}.$$

This formula (9) for geodesic curvature, which is equivalent to (4), can be verified by straightforward calculation. The details are deferred to the end of the paper. Meanwhile, we propose to use the expression (9) to give a geometric interpretation of the Schwarzian derivative of f in terms of change of curvature.

3. Geometric interpretation of the Schwarzian

In a previous paper [3] we introduced a notion of Schwarzian derivative for harmonic mappings, generalizing the classical formula

$$S(f) = \left(\frac{f''}{f'}\right)' - \tfrac{1}{2}\left(\frac{f''}{f'}\right)^2$$

for locally univalent analytic functions. If $f = h + \overline{g}$ is an orientation-preserving harmonic mapping with dilatation $g'/h' = q^2$ and conformal factor $e^{\sigma} = |h'|+|g'|$, the Schwarzian of f is defined by

$$S(f) = 2\left(\sigma_{zz} - {\sigma_z}^2\right).$$

In terms of the analytic functions h and q, the formula becomes

$$S(f) = S(h) + \frac{2\overline{q}}{1+|q|^2}\left(q'' - \frac{q'h''}{h'}\right) - 4\left(\frac{q'\,\overline{q}}{1+|q|^2}\right)^2. \tag{10}$$

Our aim is now to establish the following theorem.

Theorem 1. *Let $f(z)$ be an orientation-preserving harmonic mapping whose dilatation is the square of an analytic function, so that f lifts locally to a minimal surface defined by conformal parameters. Let C be a smooth curve in the parametric plane described by an equation $z = z(t)$, where t is arclength, and let κ be the curvature of C. Let Γ be the lift of C to the minimal surface, let s be the arclength of Γ, and let $\widehat{\kappa}$ be its geodesic curvature. Then*

$$\left|\frac{df}{dt}\right|^2 \frac{d\widehat{\kappa}}{ds} = \frac{d\kappa}{dt} + \mathrm{Im}\left\{ S(f)\left(\frac{dz}{dt}\right)^2 \right\}, \tag{11}$$

where $S(f)$ is the Schwarzian of f.

The relation (11), which interprets the Schwarzian geometrically in terms of curvature, is a direct generalization of a classical formula for analytic functions, whose associated minimal surface is a plane.

Proof. The point of departure is the formula (9) relating the curvatures of Γ and C. Here the canonical decomposition is $f = h + \overline{g}$ and the dilatation is $g'/h' = q^2$. Recall that $ds/dt = e^\sigma = |h'| + |g'|$.

Differentiating the left-hand side of (9), we find

$$\frac{d}{dt}\left(e^\sigma \widehat{\kappa}\right) = 2e^\sigma\, \mathrm{Re}\left\{\frac{\partial \sigma}{\partial z}\frac{dz}{dt}\right\}\widehat{\kappa} + e^{2\sigma}\frac{d\widehat{\kappa}}{ds},$$

since the chain rule takes the form

$$\frac{d\sigma}{dt} = 2\,\mathrm{Re}\left\{\frac{\partial \sigma}{\partial z}\frac{dz}{dt}\right\}.$$

In view of (6) and (8), it follows that

$$\frac{d}{dt}\left(e^\sigma \widehat{\kappa}\right) = \mathrm{Re}\left\{\left(e^\sigma \frac{h''}{h'} + 2\overline{q}q'|h'|\right)z'\right\}\widehat{\kappa} + e^{2\sigma}\frac{d\widehat{\kappa}}{ds}. \tag{12}$$

Introducing the formula (9) for $\widehat{\kappa}$ into the right-hand side of (12), we find after some manipulation

$$
\begin{aligned}
\frac{d}{dt}(e^{\sigma}\widehat{\kappa}) &= \kappa\,\mathrm{Re}\left\{\frac{h''}{h'}z'\right\} + \tfrac{1}{2}\,\mathrm{Im}\left\{\left(\frac{h''}{h'}z'\right)^2\right\} \\
&\quad + \frac{2}{1+|q|^2}\,\mathrm{Im}\left\{\frac{h''}{h'}\,\overline{q}q'(z')^2\right\} + \frac{2\kappa}{1+|q|^2}\,\mathrm{Re}\{\overline{q}q'z'\} \\
&\quad + \frac{2}{(1+|q|^2)^2}\,\mathrm{Im}\big\{(\overline{q}q'z')^2\big\} + e^{2\sigma}\frac{d\widehat{\kappa}}{ds}.
\end{aligned}
\tag{13}
$$

Next we differentiate the right-hand side of (9) and use the relations $z'' = i\kappa z'$ and $|z'| = 1$ to obtain

$$
\begin{aligned}
\frac{d}{dt}(e^{\sigma}\widehat{\kappa}) &= \frac{d\kappa}{dt} + \mathrm{Im}\left\{\left(\frac{h''}{h'}\right)'(z')^2 + \frac{h''}{h'}i\kappa z'\right\} \\
&\quad + \frac{2}{1+|q|^2}\,\mathrm{Im}\big\{\overline{q}q''(z')^2 + i\kappa\,\overline{q}q'z'\big\} - \frac{2}{(1+|q|^2)^2}\,\mathrm{Im}\big\{(\overline{q}q'z')^2\big\}\,.
\end{aligned}
\tag{14}
$$

We now compare the two expressions (13) and (14) to arrive at the formula

$$
\begin{aligned}
e^{2\sigma}\frac{d\widehat{\kappa}}{ds} &= \frac{d\kappa}{dt} + \mathrm{Im}\big\{S(h)(z')^2\big\} + \frac{2}{1+|q|^2}\,\mathrm{Im}\left\{\overline{q}\left(q'' - q'\frac{h''}{h'}\right)(z')^2\right\} \\
&\quad - \frac{4}{(1+|q|^2)^2}\,\mathrm{Im}\big\{(\overline{q}q'z')^2\big\}\,,
\end{aligned}
$$

where $S(h)$ is the Schwarzian derivative of h. Referring to the expression (10) for the Schwarzian of f, we see that the last equation reduces to

$$
e^{2\sigma}\frac{d\widehat{\kappa}}{ds} = \frac{d\kappa}{dt} + \mathrm{Im}\big\{S(f)(z')^2\big\}\,,
$$

which is equivalent to (11) since

$$
\left|\frac{df}{dt}\right|^2 = \left|\frac{df}{ds}\frac{ds}{dt}\right|^2 = \left(\frac{ds}{dt}\right)^2 = e^{2\sigma}.
$$

■

4. Curvature of image curves in the plane

Let $f = h + \overline{g}$ be a locally univalent orientation-preserving harmonic mapping in the unit disk $\mathbb{D}$. We wish to calculate the curvature of the image of a circle $|z| = r < 1$ at a given point $f(re^{i\theta})$. First note that the tangent vector is

$$
\frac{\partial f}{\partial \theta} = ire^{i\theta}h'(re^{i\theta}) - ire^{-i\theta}\,\overline{g'(re^{i\theta})}.
$$

Thus with the notation

$$\varphi = \arg \frac{\partial f}{\partial \theta} = \operatorname{Im}\left\{ \log \frac{\partial f}{\partial \theta} \right\},$$

the curvature is

$$\kappa = \frac{d\varphi}{ds} = \frac{d\varphi}{d\theta} \bigg/ \frac{ds}{d\theta},$$

so that

$$\begin{aligned} \left| \frac{\partial f}{\partial \theta} \right| \kappa &= \operatorname{Im}\left\{ \frac{\partial}{\partial \theta} \log \frac{\partial f}{\partial \theta} \right\} \\ &= \operatorname{Re}\left\{ \frac{e^{i\theta} h'(re^{i\theta}) + re^{2i\theta} h''(re^{i\theta}) + e^{-i\theta}\, \overline{g'(re^{i\theta})} + re^{-2i\theta}\, \overline{g''(re^{i\theta})}}{e^{i\theta} h'(re^{i\theta}) - e^{-i\theta}\, \overline{g'(re^{i\theta})}} \right\}. \end{aligned}$$

Further calculation leads to the formula

$$\begin{aligned} \frac{1}{r^2} \left| \frac{\partial f}{\partial \theta} \right|^3 \kappa &= \operatorname{Re}\Big\{ |h'|^2 + re^{i\theta}\, \overline{h'} h'' + re^{-3i\theta}\, \overline{h'}\, \overline{g''} - re^{3i\theta} h'' g' \\ &\qquad - |g'|^2 - re^{-i\theta} g'\, \overline{g''} \Big\} \\ &= |h'(z)|^2 \operatorname{Re}\left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} - |g'(z)|^2 \operatorname{Re}\left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} \\ &\quad + \operatorname{Re}\left\{ \frac{z^3}{|z|^2} \left(h'(z) g''(z) - h''(z) g'(z) \right) \right\}, \end{aligned} \tag{15}$$

where $z = re^{i\theta}$ and we understand that

$$|g'(z)|^2 \left(1 + \frac{zg''(z)}{g'(z)} \right) = |g'(z)|^2 + z\, \overline{g'(z)} g''(z)$$

vanishes wherever $g'(z) = 0$. Introducing the dilatation $\omega = g'/h'$, we find that

$$h'g'' - h''g' = (h')^2 \omega'$$

and

$$1 + \frac{zg''(z)}{g'(z)} = 1 + \frac{zh''(z)}{h'(z)} + \frac{z\omega'(z)}{\omega(z)}.$$

Consequently, we can write

$$\begin{aligned} \left| \frac{\partial f}{\partial \theta} \right|^3 \kappa &= |zh'(z)|^2 \left(1 - |\omega(z)|^2 \right) \operatorname{Re}\left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} \\ &\quad + \operatorname{Re}\left\{ z^3 \omega'(z) h'(z)^2 \right\} - |z\omega(z) h'(z)|^2 \operatorname{Re}\left\{ \frac{z\omega'(z)}{\omega(z)} \right\}. \end{aligned} \tag{16}$$

Suppose now that f has a smooth extension to some arc I of the unit circle, and that $|\omega(e^{i\theta})| \equiv 1$ there. Since $\omega\,\overline{\omega} = 1$, we have

$$0 = \frac{d}{d\theta}\left(\omega(e^{i\theta})\,\overline{\omega(e^{i\theta})}\right) = ie^{i\theta}\omega'(e^{i\theta})\,\overline{\omega(e^{i\theta})} - ie^{-i\theta}\omega(e^{i\theta})\,\overline{\omega'(e^{i\theta})},$$

so that $e^{i\theta}\omega'(e^{i\theta})\,\overline{\omega(e^{i\theta})}$ is real. Also, because f is orientation-preserving and so $|\omega(z)| < 1$ in $\mathbb{D}$, it follows from Hopf's lemma (see [5], p. 116) that

$$\frac{\partial}{\partial r}\left(|\omega(re^{i\theta})|^2\right) > 0 \qquad \text{for } r = 1,$$

which shows that $e^{i\theta}\omega'(e^{i\theta})\,\overline{\omega(e^{i\theta})} > 0$. Therefore,

$$e^{i\theta}\omega'(e^{i\theta})\,\overline{\omega(e^{i\theta})} = |\omega'(e^{i\theta})| > 0, \tag{17}$$

and we deduce from (16) that

$$\left|\frac{\partial f}{\partial\theta}\right|^3 \kappa = \mathrm{Re}\left\{e^{3i\theta}\omega'(e^{i\theta})h'(e^{i\theta})^2\right\} - |\omega'(e^{i\theta})||h'(e^{i\theta})|^2.$$

In particular, $\kappa \le 0$ on the arc I where $|\omega(e^{i\theta})| = 1$. Equality occurs only where

$$e^{3i\theta}\omega'(e^{i\theta})h'(e^{i\theta})^2 = |\omega'(e^{i\theta})||h'(e^{i\theta})|^2,$$

or by (17)

$$e^{3i\theta}\omega'(e^{i\theta})h'(e^{i\theta}) = e^{i\theta}\omega'(e^{i\theta})\,\overline{\omega(e^{i\theta})}\,\overline{h'(e^{i\theta})},$$

which reduces to the statement that

$$\frac{d}{d\theta}\left(h(e^{i\theta}) + \overline{g(e^{i\theta})}\right) = 0.$$

Therefore, if $df/d\theta \ne 0$ on an arc where $|\omega(e^{i\theta})| = 1$, then $\kappa < 0$. In other words, the boundary is *concave* along the image arc. This phenomenon is implicit in work of Hengartner and Schober [8]. An explicit proof, also based on Hopf's lemma, was given by Duren and Khavinson [6], and Bshouty and Hengartner [1] gave another proof, but the above argument is more direct. The result can be summarized as follows.

Theorem 2. *Let f be a locally univalent harmonic mapping in the unit disk, and suppose it has a C^2 extension to some arc I of the unit circle, with dilatation of unit modulus and $df/d\theta \ne 0$ on I. Then f maps I onto an arc of negative curvature.*

In view of Theorem 2, it is clear that if f is a univalent harmonic mapping of the disk with smooth extension to the boundary, and if its dilatation has unit modulus on the whole unit circle, then f must have stationary points. In other words, it is necessarily true that $df/d\theta = 0$ at some points of the circle.

To give an illustration, let Φ be a conformal mapping of $\mathbb{D}$ onto a domain convex in the horizontal direction, whose boundary is an analytic Jordan curve. Then Φ has an analytic continuation to the closed disk with $\Phi'(e^{i\theta}) \ne 0$. Consider now the harmonic shear $f = h + \overline{g}$ of Φ with dilatation $\omega = g'/h'$ prescribed to be a finite Blaschke product. Then f is a univalent harmonic mapping of $\mathbb{D}$ onto a domain Ω convex in the horizontal direction, with $h - g = \Phi$. (The shear construction was introduced by Clunie and Sheil-Small [2]. See also [5], Section 3.4.) If ω is a Blaschke product of order n, then $|\omega(e^{i\theta})| \equiv 1$ and $\omega(e^{i\theta}) = 1$ at precisely n distinct points. In view of the relation

$$\Phi' = h' - g' = (1-\omega)h', \tag{18}$$

we conclude that h is analytic in $\overline{\mathbb{D}}$ and $h'(e^{i\theta}) \neq 0$, except where $\omega(e^{i\theta}) = 1$. The relation (18) also shows that h has a logarithmic singularity at points where $\omega(e^{i\theta}) = 1$. Away from these n points, a calculation gives

$$\begin{aligned}\frac{df}{d\theta} &= ie^{i\theta}h'(e^{i\theta}) - ie^{-i\theta}\,\overline{g'(e^{i\theta})} \\ &= \frac{ie^{i\theta}\Phi'(e^{i\theta})}{1-\omega(e^{i\theta})} - \frac{ie^{-i\theta}\,\overline{\omega(e^{i\theta})}\,\overline{\Phi'(e^{i\theta})}}{1-\overline{\omega(e^{i\theta})}} \\ &= \frac{i}{1-\omega(e^{i\theta})}\left(e^{i\theta}\Phi'(e^{i\theta}) + e^{-i\theta}\,\overline{\Phi'(e^{i\theta})}\right),\end{aligned}$$

since $|\omega(e^{i\theta})| = 1$. This shows that $df/d\theta = 0$ precisely at those points $e^{i\theta}$ where $ie^{i\theta}\Phi'(e^{i\theta})$ is real; *i.e.*, where the tangent vector is horizontal, if $\omega(e^{i\theta}) \neq 1$ at those points. Consequently, f extends smoothly to the open boundary arcs between points where $\omega(e^{i\theta}) = 1$, and $df/d\theta \neq 0$ except at points where the curve $\Phi(e^{i\theta})$ has a horizontal tangent vector. The behavior of $df/d\theta$ is indeterminate at points where both $\omega(e^{i\theta}) = 1$ and the tangent vector is horizontal. The points of either type divide the circle into a finite number of arcs, each of which f maps to a concave boundary arc of Ω.

For a specific example, consider the harmonic shear f_n of the identity map $\Phi(z) = z$ with dilatation $\omega(z) = z^n$, where n is a positive integer. Then $\omega(e^{i\theta}) = 1$ at the nth roots of unity $e^{2\pi ik/n}$, for $k = 1, 2, \dots, n$; whereas the tangent vector of the curve $\Phi(e^{i\theta})$ is horizontal at $e^{i\theta} = \pm i$. The points $\pm i$ are nth roots of unity when $n = 4, 8, 12, \dots$. For all other integers n, the harmonic mapping f_n sends each of the $n+2$ arcs on the unit circle between successive roots of unity or points $\pm i$ onto a concave boundary arc of its image. An explicit formula for f_n is found to be

$$f_n(z) = -\overline{z} - \frac{2}{n}\,\mathrm{Re}\left\{\sum_{k=1}^{n} \alpha^k \log\bigl(1 - \alpha^{-k}z\bigr)\right\}, \qquad \alpha = e^{2\pi i/n}.$$

Figure 1, produced by Mathematica, shows the images under f_n of equally spaced concentric circles and radial lines in the unit disk, for $n = 1, 2, \dots, 8$. Note that the points $e^{2\pi ik/n}$ are mapped to infinity, as predicted, whereas the points $\pm i$ are mapped to finite cusps. Exceptions occur when n is a multiple of 4 and the two sets of points overlap. Then the boundary function $f_n(e^{i\theta})$ has

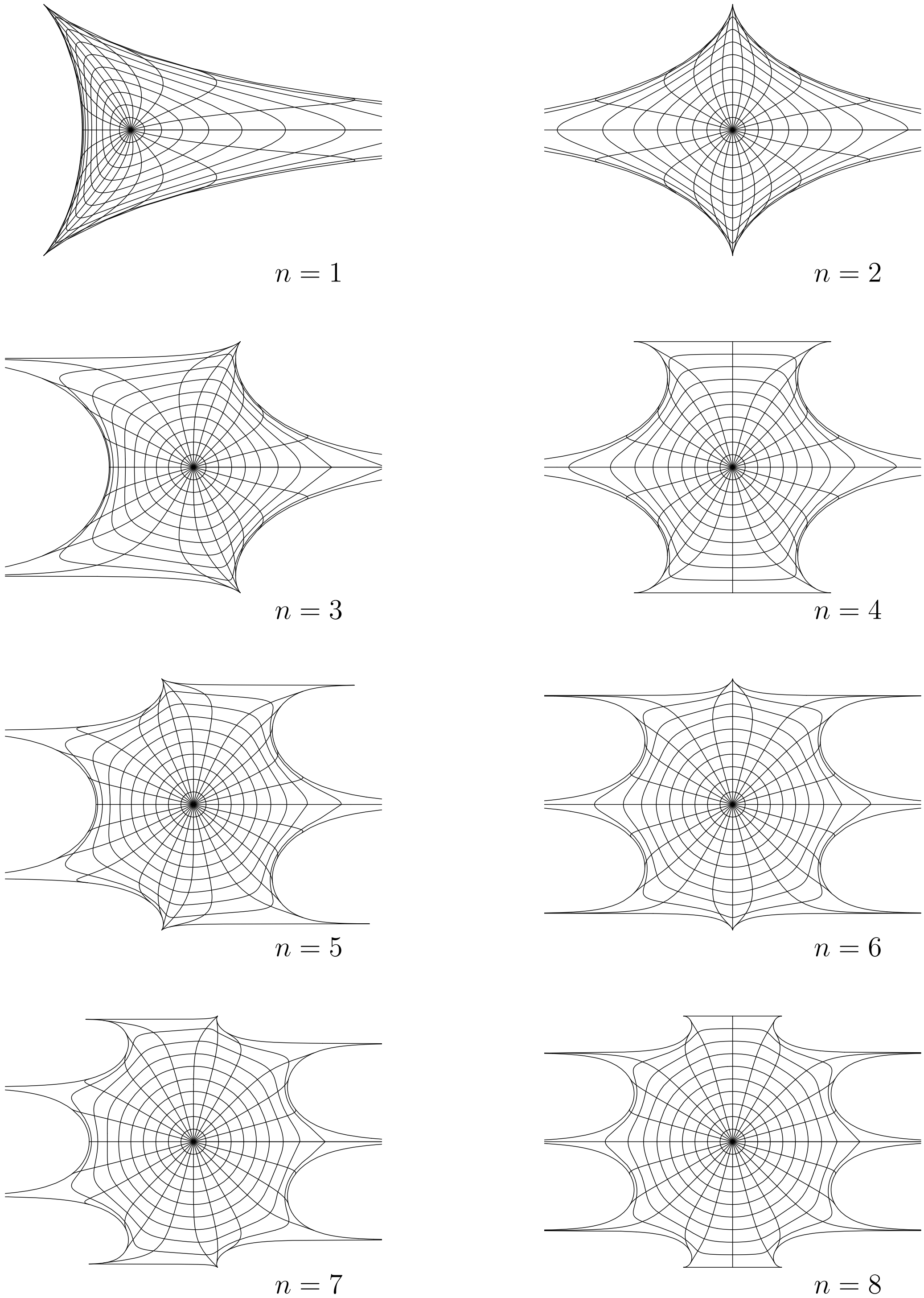

FIGURE 1. Harmonic shear of identity with dilatation z^n.

a horizontal jump at $\theta = \pm\pi/2$. Greiner [7] studied the same example from a different viewpoint and found the magnitude of the jump to be $2\pi/n$. The phenomenon is also implicit in a formula given by Hengartner and Schober [8] (*cf.* [5], Section 7.4).

A simpler example is given by the function

$$F_n(z) = z + \frac{1}{n+1}\,\overline{z}^{n+1}, \qquad n = 1, 2, \ldots,$$

which maps the unit disk onto a domain bounded by a hypocycloid of $n+2$ cusps, inscribed in the circle $|w| = (n+2)/(n+1)$.

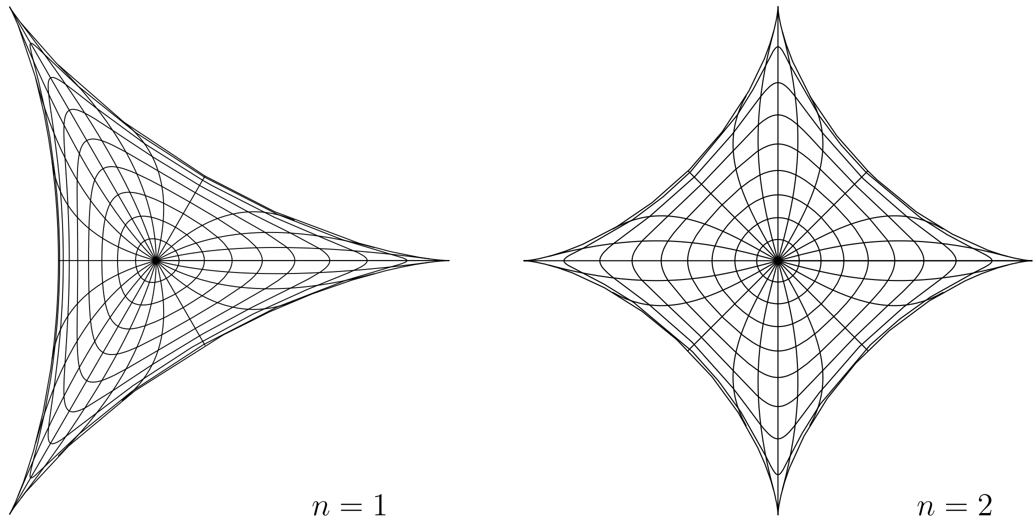

FIGURE 2. Harmonic mapping $F_n(z) = z + \frac{1}{n+1}\,\overline{z}^{n+1}$.

Figure 2 shows the images of F_1 and F_2. Here $h(z) = z$ and $\omega(z) = z^n$, so that equation (16) reduces to the simple formula

$$\kappa = -\frac{n}{2\sqrt{2}}\big(1 - \cos(n+2)\theta\big)^{-1/2},$$

since $|df/d\theta| = ds/d\theta = \sqrt{2}(1 - \cos(n+2)\theta)^{1/2}$. In general, the Gauss–Bonnet formula says that

$$\int_{\partial\Omega} \kappa\, ds + \sum_{k=1}^{n+2} \beta_k = 2\pi,$$

where β_k are the changes in the argument of the tangent vector at the $n+2$ vertices. For the particular example at hand,

$$\int_{\partial\Omega} \kappa\, ds = -\frac{n}{2}\int_0^{2\pi} d\theta = -n\pi,$$

and $\beta_k = \pi$ at each cusp.

5. Convexity and starlikeness

It is well known that convexity is a hereditary property for conformal mappings. In other words, if f is analytic and univalent in $\mathbb{D}$ and maps it onto a convex domain, then the image of every subdisk $|z| < r < 1$ is also convex. However, the hereditary property does not generalize to harmonic mappings. If f is a univalent harmonic mapping of $\mathbb{D}$ onto a convex domain, then the image of the disk $|z| < r$ is convex for each radius $r \le \sqrt{2} - 1$ but not necessarily for any radius in the interval $\sqrt{2} - 1 < r < 1$. The function

$$L(z) = \mathrm{Re}\{\ell(z)\} + i\,\mathrm{Im}\{k(z)\} = \mathrm{Re}\left\{\frac{z}{1-z}\right\} + i\,\mathrm{Im}\left\{\frac{z}{(1-z)^2}\right\} \tag{19}$$

provides a harmonic mapping of the disk onto the half-plane $\mathrm{Re}\{w\} > -1/2$, yet the image of the disk $|z| \le r$ fails to be convex for every r in the interval $\sqrt{2} - 1 < r < 1$.

In the same sense, starlikeness is a hereditary property for conformal mappings. Thus if f is analytic and univalent in $\mathbb{D}$ with $f(0) = 0$, and if f maps $\mathbb{D}$ onto a domain that is starlike with respect to the origin, then the image of every subdisk $|z| < r < 1$ is also starlike with respect to the origin. This means that every point of the range can be connected to the origin by a radial line that lies entirely in the region. Again, this hereditary property does not generalize to harmonic mappings.

A harmonic mapping of the unit disk will be called *fully convex* if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a convex curve. Similarly, a harmonic mapping f with $f(0) = 0$ is said to be *fully starlike* if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve that bounds a domain starlike with respect to the origin. In particular, $f(z) \neq 0$ for $0 < |z| < 1$. According to the Radó–Kneser–Choquet theorem (see *e.g.* [5], Section 3.1), a fully convex harmonic mapping is necessarily univalent in $\mathbb{D}$. However, we shall see that a fully starlike mapping need not be univalent.

The following theorem gives an analytic description of the fully convex and fully starlike mappings. It provides a harmonic generalization of the standard criteria

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad \text{and} \qquad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$$

for an analytic function f to be convex and starlike, respectively.

Theorem 3. *Let $f = h+\overline{g}$ be a locally univalent orientation-preserving harmonic mapping of $\mathbb{D}$. Then f is fully convex if and only if*

$$|zh'(z)|^2 \operatorname{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} \geq |zg'(z)|^2 \operatorname{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} + \operatorname{Re}\left\{z^3 \left[h''(z)g'(z) - h'(z)g''(z)\right]\right\} \tag{20}$$

for all $z \in \mathbb{D}$. If $f(0) = 0$, then f is fully starlike if and only if $f(z) \neq 0$ for $0 < |z| < 1$ and

$$|h(z)|^2 \operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} \geq |g(z)|^2 \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} + \operatorname{Re}\{z\left[h(z)g'(z) - h'(z)g(z)\right]\} \tag{21}$$

for all $z \in \mathbb{D}$.

Proof. Suppose first that f is fully convex. Then f maps each circle $|z| = r < 1$ onto a curve with curvature $\kappa(re^{i\theta}) \geq 0$. According to the formula (15), this condition reduces to (20). Conversely, if (20) holds, then f maps each circle $|z| = r < 1$ onto a curve with $\kappa(re^{i\theta}) \geq 0$. Since f is univalent in a neighborhood of the origin, the total curvature $\int_0^{2\pi} \kappa(re^{i\theta})d\theta$ is equal to 2π for each r sufficiently small. But the integral varies continuously with r and is an integer multiple of 2π, so the total curvature is equal to 2π for every r. In other words, f maps each circle $|z| = r < 1$ in a one-to-one manner onto a convex curve, and so f is fully convex.

Suppose next that f is fully starlike. Then $\frac{\partial}{\partial\theta} \arg(f(re^{i\theta})) \geq 0$ for $0 < r < 1$. This can be written as $\frac{\partial}{\partial\theta} \operatorname{Im}\{\log f(re^{i\theta})\} \geq 0$, or

$$\operatorname{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \geq 0, \qquad z = re^{i\theta}.$$

Equivalently,

$$\operatorname{Re}\left\{\left(zh'(z) - \overline{zg'(z)}\right)\left(\overline{h(z)} + g(z)\right)\right\} \geq 0,$$

which reduces to the relation (21). Conversely, if $f(z) \neq 0$ for $z \neq 0$ and (21) holds, then $\arg(f(re^{i\theta}))$ is nondecreasing along each circle $|z| = r$. The total change of argument is a continuous function of r and is equal to 2π for small r by the local univalence of f, so the image under f of each circle $|z| = r < 1$ is a curve of nondecreasing argument that winds exactly once about the origin. This says that each image curve bounds a starlike domain, so f is fully starlike. ∎

We now turn to the harmonic analogue of Alexander's theorem. The following theorem adapts an observation by Sheil-Small [11] to the context of fully convex and fully starlike mappings.

Theorem 4. *Let h, g, H and G be analytic functions in the unit disk, related by*

$$zH'(z) = h(z), \qquad zG'(z) = -g(z), \qquad z \in \mathbb{D}.$$

Then $f = h + \overline{g}$ is fully starlike if and only if $F = H + \overline{G}$ is fully convex.

Proof. If F is fully convex, it is locally univalent, so that $|H'(z)| \neq |G'(z)|$ in $\mathbb{D}$. This implies that $f(0) = 0$ and $f(z) \neq 0$ for $z \neq 0$. Without loss of generality, we may assume that F is orientation-preserving. If (20) holds for $F = H + \overline{G}$, then a simple calculation shows that (21) holds for $f = h + \overline{g}$. Thus f is fully starlike.

Conversely, if f is fully starlike, then F is locally univalent and may be taken to be orientation-preserving. Then f satisfies (21) and so F satisfies (20), which implies that F is fully convex. ∎

The harmonic half-plane mapping (19) has the form $L(z) = H(z) + \overline{G(z)}$ with

$$H(z) = \tfrac{1}{2}\left(\ell(z) + k(z)\right) = \frac{z - \frac{1}{2}z^2}{(1-z)^2},$$

$$G(z) = \tfrac{1}{2}\left(\ell(z) - k(z)\right) = \frac{-\frac{1}{2}z^2}{(1-z)^2}.$$

Further calculation shows that

$$h(z) = zH'(z) = \frac{z}{(1-z)^3}, \qquad g(z) = -zG'(z) = \frac{z^2}{(1-z)^3},$$

so that

$$h'(z) = \frac{1+2z}{(1-z)^4}, \qquad g'(z) = \frac{2z + z^4}{(1-z)^4}.$$

The mapping $f = h + \overline{g}$ has a Jacobian

$$J(z) = |h'(z)|^2 - |g'(z)|^2 = |1-z|^{-8}\left(|1+2z|^2 - |z|^2|1+z|^2\right).$$

Straightforward analysis shows that $J(z) > 0$ for $|z| < 2 - \sqrt{3}$ but $J(-2+\sqrt{3})$ vanishes. Thus $f = h + \overline{g}$ is not univalent in any disk $|z| < r$ with $r > 2 - \sqrt{3}$. On the other hand, $L(z)$ is known to map each of the circles $|z| = r \leq \sqrt{2} - 1$ onto a convex curve, whereas the image is not convex for $\sqrt{2} - 1 < r < 1$. (See for instance [5], Section 3.5.) Since $2 - \sqrt{3} < \sqrt{2} - 1$, it follows that the dilation

$$\widetilde{F}(z) = F((\sqrt{2}-1)z) = \widetilde{H}(z) + \overline{\widetilde{G}(z)}$$

is a fully convex mapping whose companion $\widetilde{f}(z) = \widetilde{h}(z) + \overline{\widetilde{g}(z)}$ defined by

$$\widetilde{h}(z) = z\widetilde{H}'(z), \qquad \widetilde{g}(z) = -z\widetilde{G}'(z)$$

is fully starlike but is not univalent in any disk $|z| < r$ of radius larger than

$$\frac{2-\sqrt{3}}{\sqrt{2}-1} = 0.646\ldots.$$

In particular, this shows that a fully starlike mapping need not be univalent.

6. Calculation of geodesic curvature

Our final task is to verify the formula (9) for geodesic curvature that was used in the proof of Theorem 1. It is convenient to consider the equivalent formula (7). Introducing the expression (3) for $\widehat{\kappa}$ into (7), we see that it suffices to prove

$$\varphi''(t)\cdot\mathbf{n} = e^{2\sigma}\kappa + e^{\sigma}\,\mathrm{Im}\Big\{\Big(\frac{h''}{h'}|h'| + \frac{g''}{g'}|g'|\Big)\,z'\Big\}. \tag{22}$$

Differentiation of (1) gives

$$\varphi''(t) = \langle h''(z')^2 + h'z'' + \overline{g''(z')^2 + g'z''}, 2\,\mathrm{Im}\big\{(h'q)'(z')^2 + h'qz''\big\}\rangle.$$

Hence, using the expression (2) for $\mathbf{n}$ and recalling that $|z'| = 1$, we find after rearrangement of terms that $\varphi''(t)\cdot\mathbf{n} = A + B$, where

$$A = \mathrm{Re}\big\{-iz''\left[(|h'|^2 + |g'|^2)\,\overline{z'} - 2h'g'z'\right]\big\} + 4\,\mathrm{Im}\{h'qz''\}\,\mathrm{Re}\{h'qz'\}\,,$$

$$B = \mathrm{Im}\big\{\big(h''\,\overline{h'} + g''\,\overline{g'}\big)\,z' - (h''g' + h'g'')\,(z')^3\big\} + 4\,\mathrm{Im}\big\{(h'q)'(z')^2\big\}\,\mathrm{Re}\{h'qz'\}\,.$$

Now introduce the relation $z'' = i\kappa z'$ to obtain after further reduction

$$A = \kappa\left(|h'|^2 + |g'|^2 - 2\,\mathrm{Re}\big\{h'g'(z')^2\big\} + 4\,\mathrm{Re}\{h'qz'\}^2\right).$$

Observe that $(h'qz')^2 = h'g'(z')^2$ and use the identity $\mathrm{Re}\{\alpha^2\} = \mathrm{Re}\{\alpha\}^2 - \mathrm{Im}\{\alpha\}^2$ to write

$$\begin{aligned} A &= \kappa\left(|h'|^2 + |g'|^2 + 2\,\mathrm{Re}\{h'qz'\}^2 + 2\,\mathrm{Im}\{h'qz'\}^2\right) \\ &= \kappa\left(|h'|^2 + |g'|^2 + 2|h'qz'|^2\right) \\ &= \kappa\left(|h'|^2 + |g'|^2\right)^2 \\ &= e^{2\sigma}\kappa. \end{aligned}$$

Therefore, it remains to show that B is equal to the second term on the right-hand side of (22). But $(h'q)^2 = h'g'$, so

$$2(h'q)(h'q)' = h''g' + h'g'', \qquad 2\overline{(h'q)}(h'q)' = |h'g'|\left(\frac{h''}{h'} + \frac{g''}{g'}\right).$$

Hence we can write

$$\begin{aligned} 4\,\mathrm{Im}\big\{(h'q)'(z')^2\big\}\,\mathrm{Re}\{h'qz'\} &= 2\,\mathrm{Im}\big\{(h'q)'(z')^2\left(h'qz' + \overline{h'qz'}\right)\big\} \\ &= \mathrm{Im}\left\{(h''g' + h'g'')(z')^3 + |h'g'|\left(\frac{h''}{h'} + \frac{g''}{g'}\right)z'\right\}, \end{aligned}$$

which implies that

$$B = \mathrm{Im}\left\{\big(h''\,\overline{h'} + g''\,\overline{g'}\big)\,z' + |h'g'|\left(\frac{h''}{h'} + \frac{g''}{g'}\right)z'\right\}. \tag{23}$$

A simple calculation now shows that

$$B = e^{\sigma}\,\mathrm{Im}\left\{\left(\frac{h''}{h'}|h'| + \frac{g''}{g'}|g'|\right) z'\right\},$$

since the last expression also reduces to the form (23). This proves (22) and concludes the derivation of the formula (7) for geodesic curvature.

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