John Domains and a Univalence Criterion of Ahlfors

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1 Introduction: Ahlfors's Univalence Criterion

In this note we special mapping properties of an analytic function f satisfying any of the 1-parameter families of bounds on the Schwarzian dervative Sf of a type first considered by Ahlfors:

$$\left| Sf(z) - \frac{2t(1-t)\bar{z}^2}{(1-|z|^2)^2} \right| \leq \frac{2t}{(1-|z|^2)^2}\,, \tag{1.1}$$

for $1 \leq t \leq 2$, and

$$\left| Sf(z) - \frac{2t(1-t)\bar{z}^2}{(1-|z|^2)^2} \right| \leq \frac{2t}{(1-|z|^2)^2} + \frac{2\pi}{(1-|z|^2)^{2t}} \left(\frac{\Gamma(\frac{3}{2}-t)}{\Gamma(1-t)} \right)^2 , \tag{1.2}$$

for $0 \leq t < 1$. Here f is analytic and locally univalent in the unit disk $\mathbf{D}$ and

$$Sf = \left(\frac{f''}{f'}\right) - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

Recall that

$$S(T \circ f) = Sf$$

when T is a Möbius transformation.

The inequalities (1.1) and (1.2) are univalence criteria; a function satisfying either is univalent in the disk. Furthermore, as was shown in [3] for more general criteria, any such function has a continuous extension to $\overline{\mathbf{D}}$. Taking $t = 0$ and 1, respectively, we obtain well known and well studied conditions of Nehari,

$$|Sf(z)| \leq \frac{\pi^2}{2} \quad \text{and} \quad |Sf(z)| \leq \frac{2}{(1-|z|^2)^2}.$$

We will exlcude the case $t = 1$ from our analysis except for a few comments later on. For the other values of t the conditions were derived originally by Ahlfors in [1]. He allowed the parameter t to be complex, but his condition did not include the second term on the right hand side of (1.2). We explain this briefly at the end of this section.

Next, a bounded, simply connected domain Ω is called a *John domain* if there is a positive constant a such that for every crosscut C of Ω the inequality

$$\operatorname{diam} H \leq a \operatorname{diam} C,$$

holds for a component H of $\Omega \backslash C$, where the condition is on the euclidean diameter; see [7]. A John domain is not necessarily a Jordan domain.

We shall prove the following theorem.

Theorem 1 *Let $1 < t \leq 2$. If f satisfies (1.1) and $f(\mathbf{D})$ is bounded, then $f(\mathbf{D})$ is a John domain.*

This is surprising, but with the proper set-up the analysis becomes quite simple.

More is true for $0 \leq t < 1$ and the criterion (1.2). First, recall that a quasidisk is the image of $\mathbf{D}$ under a quasiconformal mapping of the sphere. A bounded quasidisk is a John domain, but not conversely. Next, if $0 \leq t < 1$ and f satisfies (1.2) then

$$(1-|z|^2)^2|Sf(z)| \leq 2t(1-t)|z|^2 + 2t + O((1-|z|^2)^{2(1-t)}),$$

and hence

$$\limsup_{|z|\to 1}(1-|z|^2)^2|Sf(z)| \leq 2t(2-t) < 2.$$

A theorem of Gehring and Pommerenke in [5] now allows us to conclude that $f(\mathbf{D})$ is a quasidisk as long as $f(\mathbf{D})$ is a Jordan domain.

To go further we adopt the terminology in [3] and say that a function F satisfying a univalence criterion (such as the ones we are considering, but even more generally) is an *extremal function* for the criterion if $F(\mathbf{D})$ is *not* a Jordan domain. An extremal function for the case $t = 0$ is $F(z) = \tan(\frac{\pi}{2}z)$; this is the unique extremal up to rotation. For $0 < t < 1/2$, there are *no* extremal functions, *i.e.*, all images are Jordan domains. For $1/2 \leq t < 1$ extremals are given by

$$F(z) = \frac{\delta}{\pi}\tan\left(\frac{\pi}{\delta}\int_0^z \frac{d\zeta}{(1-z^2)^t}\right), \tag{1.3}$$

where

$$\delta = \delta(t) = 2\int_0^1 \frac{dx}{(1-x^2)^t} = \sqrt{\pi}\frac{\Gamma(1-t)}{\Gamma(\frac{3}{2}-t)}. \tag{1.4}$$

Once again, in this case these are the unique extremals up to rotation. For these facts see [2] and [3].

We summarize these remarks as a theorem.

Theorem 2 *If $0 \leq t < 1$ and f satisfies (1.2) then the image $f(\mathbf{D})$ is a quasidisk unless f is an extremal function. If $0 < t < 1/2$ the image is a quasidisk.*

We are grateful to the referee for pointing out this simple application of the Gehring-Pommerenke theorem as a strengthening of Theorem 1 for t in this range. Previously we had shown, by arguments similar to those in the next section, that the images are John donains (in all cases, including the extremals).

We also call attention to a phenomenon for the case $t = 1$ (Nehari's $2/(1-|z|^2)^2$ - criterion). The John condition forbids outward cusps and is, in that sense, 'half of' what it takes for a domain to be a quasidisk. In [4] it was shown that if f satisfies (1.1) for $t = 1$ and if $f(\mathbf{D})$ is a John domain, then it is a quasidisk. We do not know if this holds for the parametrized family of criteria in Theorem 1, that is if all the images other than the extremals are quasidisks in this case too.

We conclude this introduction with a few remarks about Ahlfors's criteria in a broader context. It is now understood that the conditions (1.1) and (1.2) correspond to particular cases of a very general univalence criterion in [6] for conformal mappings of Riemannian manifolds, applied in this case to $\mathbf{D}$ with the 1-parameter family of metrics

$$g_t = \frac{|dz|^2}{(1-|z|^2)^{2t}}. \tag{1.5}$$

Both the Gaussian curvature of the metric and the metric diameter of the disk enter in applying the general theorem. The curvature of g_t is always non-positive, though not constant. If $t \geq 1$ then g is complete, while if $t < 1$, $\mathbf{D}$ has finite diameter equal to $\delta(t)$ in (1.4), which is how the gamma functions come into (1.2). Note that $\delta(t) \to \infty$ as t increases to 1, and (1.1) and (1.2) coincide.

2 Proof of Theorem 1: Convexity and Extremal Functions

We will use a characterization of John domains proved in [4], namely that a simply connected domain $f(\mathbf{D})$ is John if and only if there exist $r_0, \sigma \in (0,1)$ and $\beta > 1$ such that

$$\frac{(1-r^2)|f'(r\zeta)|}{(1-s^2)|f'(s\zeta)|} \geq \beta \tag{2.1}$$

for all $|\zeta| = 1$, $r_0 \leq r < 1$ and

$$s = \frac{r+\sigma}{1+r\sigma}.$$

Naturally, this is a condition on a neighborhood of the boundary.

We will also appeal to some of the results in [3] applied to Ahlfors's criteria. The key ideas have to do with convexity.

Let f be a bounded function satisfying (1.1). Compose f with a Möbius transformation of the range so that

$$f(z) = \frac{1}{z} + b_0 + b_1 z + \cdots, \tag{2.2}$$

still calling the normalized function f. Composition with a Möbius transformation does not change the Schwarzian, and we are going to show that the normalized function satisfies (2.1) near $|z| = 1$. Since we started with a bounded function, applying the inverse of the Möbius transformation used to get (2.2) will not invalidate the John condition (2.1).

Let $|\zeta| = 1$ and for $0 \leq r < 1$ define

$$u(r) = \frac{1}{\sqrt{(1-r^2)^t |f'(r\zeta)|}}\,.$$

The normalization on f implies that

$$u(0) = 0, \quad u'(0) = 1. \tag{2.3}$$

¿From the convexity results in [3] it follows that

$$((1-r^2)^t u'(r))' \geq 0. \tag{2.4}$$

Thus u satisfies

$$(pu')' \geq 0$$

with the initial conditions $u(0) = 0$ and $u'(0) = 1$, where

$$p(r) = (1-r^2)^t.$$

The differential inequality (2.4) can be checked directly, but it arises from (1.1) (and more generally in [3]) by evaluating the Hessian of u along a radius and using the fact that a radius is a geodesic for the metric $|dz|/(1-|z|^2)^t$. The factor of $(1-r^2)^t$ comes from giving the radius its arclength parametrization for this metric.

Next, an extremal function for (1.1) is

$$F(z) = \int_0^z \frac{d\zeta}{(1-\zeta^2)^t}. \tag{2.5}$$

For this extremal $F(1) = F(-1) = \infty$, and F maps $[0, 1)$ onto $[0, \infty)$ with $F(0) = 0$ and $F'(0) = 1$. Since $F'(r) = 1/p(r)$, we may write

$$(pF')' = 0, \tag{2.6}$$

We now have a differential equation and inequality, (2.6) and (2.4), and it is then trivial to check that

$$w = u \circ F^{-1}$$

is a convex function. (We will state a more general, but no more difficult version of this observation at the end of this section.)

We make use of this convexity to comapare u to the extremal function F. Let $y_2 \geq y_1 > 0$. From the convexity and the initial conditions we have

$$w(y_2) - w(y_1) \geq w'(y_1)(y_2 - y_1) \quad \text{and} \quad w(y_1) \leq w'(y_1)y_1,$$

and hence

$$\frac{w(y_2)}{w(y_1)} \geq \frac{y_2}{y_1}.$$

Now let $r = F^{-1}(y_1)$ and $s = F^{-1}(y_2)$. Then

$$\frac{u(s)}{u(r)} \geq \frac{F(s)}{F(r)}.$$

We shall estimate the right hand side.

Assume now that $t > 1$, as in the hypotheses of Theorem 1. We have

$$\begin{aligned}
\frac{F(s)}{F(r)} &= \frac{\displaystyle\int_0^s \frac{dx}{(1-x^2)^t}}{\displaystyle\int_0^r \frac{dx}{(1-x^2)^t}} \geq \frac{1}{2^t} \frac{\displaystyle\int_0^s \frac{dx}{(1-x)^t}}{\displaystyle\int_0^r \frac{dx}{(1-x)^t}} \\
&= \frac{1}{2^t}\left(\frac{1-r}{1-s}\right)^\mu \frac{1-(1-s)^\mu}{1-(1-r)^\mu} \geq \frac{1}{2^t}\left(\frac{1-r}{1-s}\right)^\mu,
\end{aligned}$$

where $\mu = t - 1$. Thus

$$\frac{u(s)}{u(r)} \geq \frac{1}{2^t}\left(\frac{1-r}{1-s}\right)^\mu,$$

which implies that

$$\frac{(1-r^2)|f'(r\zeta)|}{(1-s^2)|f'(s\zeta)|} \geq \left(\frac{1}{2^t}\right)^2 \left(\frac{1-s^2}{1-r^2}\right)^\mu \left(\frac{1-r}{1-s}\right)^{2\mu} \geq \left(\frac{1}{2^t}\right)^2 \left(\frac{1-r}{1-s}\right)^\mu.$$

If

$$s = \frac{r+\sigma}{1+r\sigma}$$

then

$$\left(\frac{1-r}{1-s}\right)^\mu = \left(\frac{1+r\sigma}{1-\sigma}\right)^\mu \geq \frac{1}{(1-\sigma)^\mu},$$

which shows that

$$\frac{(1-r^2)|f'(r\zeta)|}{(1-s^2)|f'(s\zeta)|}$$

can be made bigger than any fixed β provided σ is chosen sufficiently close to 1.

This completes the proof of the theorem.

Remark The argument above uses a special case of a statement on the 'relative convexity' of solutions of a differential equation and inequality. We have found this useful here and on other occasions and we formulate it as follows.

Lemma [Relative Convexity] *Let u, v, $p > 0$ and q be defined in $[0,1)$ and suppose that*

$$(pu')' + qu \geq 0, \tag{2.7}$$

and that

$$(pv')' + qv = 0. \tag{2.8}$$

Then the function

$$w = (\frac{u}{v}) \circ F^{-1}$$

is convex, where F is defined by $F' = 1/pv^2$.

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