Delay-Spread Distribution for Multimode Fiber With Strong Mode Coupling

Keang-Po Ho and Joseph M. Kahn

Abstract—**In the strong mode coupling regime, the delay spread of multimode fiber is statistically the same as the difference between the maximum and minimum eigenvalues of a Gaussian unitary ensemble. We study the delay-spread distribution using three methods: 1) numerical evaluation of the Fredholm determinant; 2) numerical integration based on the Andréief identity; and 3) approximation based on the Tracy–Widom distribution. Results obtained using the Fredholm determinant and the Andréief identity are virtually indistinguishable. The approximation based on the Tracy–Widom distribution is sufficiently accurate for most engineering purposes when the number of modes is at least 12. In a digital equalizer, a memory length of four to five times the group-delay standard deviation is sufficient to ensure that the delay spread will exceed the equalizer memory length with a probability of less than 10^{-4}–10^{-6}.**

Index Terms—**Delay spread, mode-division multiplexing, multimode fiber, random matrices.**

I. Introduction

When mode-division multiplexing is used in multimode fiber [1]–[5], system throughput ideally increases in proportion to the number of propagating modes. The group velocities of various modes are generally slightly different, leading to modal dispersion [6]–[9]. While modal dispersion is not fundamentally a performance-limiting factor, the inter-modal group delay spread affects the complexity of digital signal processing used to compensate modal dispersion and separate multiplexed signals [10]. Strong mode coupling can reduce the delay spread, making it proportional to the square-root of the fiber length [8], [9], thereby minimizing signal processing complexity [10]. Strong mode coupling can also reduce the effects of mode-dependent gains and losses [11], [12] and provides frequency diversity, so the system outage capacity approaches the average capacity [13].

When a system is in the strong mode coupling regime, the statistics of the modal group delays can be described by the eigenvalues of a zero-trace Gaussian unitary ensemble (GUE) [9]. The delay spread is the difference between the maximum and minimum group delays [10], [14], and is equivalent to the difference between the maximum and minimum eigenvalues of the zero-trace GUE. Aside from a scale factor close to unity, the statistics of the difference between the maximum and minimum eigenvalues are the same for GUEs with and without the zero-trace constraint.

The statistics of the delay spread were estimated empirically through numerical simulation in [10], [14], and these statistics were used to study signal processing complexity in [10]. In this letter, the statistics of the delay spread, or the difference between the maximum and minimum eigenvalues of a GUE, are studied by three methods. The first method is based on numerical evaluation of the Fredholm determinant, which is equivalent to the solution of an integral equation. The second method uses the Andréief identity, such that only one-dimensional numerical integration is required. These two methods, which are exact in principle, yield almost identical numerical results, and match the numerical simulations in [10]. The third method approximates the statistics of the delay spread using the Tracy–Widom distribution, under the assumption that the maximum and minimum eigenvalues are independent of each other. It yields results close to exact when the number of modes is at least 12.

The remainder of this letter is organized as follows. Section II describes the three methods and presents the delay-spread statistics they yield. Sections III and IV provide discussion and conclusions, respectively.

II. Delay-Spread Distribution

We consider a fiber supporting D propagating modes, including spatial and polarization degrees of freedom. The group delays of the multimode fiber are statistically the same as the eigenvalues of a $D \times D$ zero-trace GUE $\mathbf{G}$ [9]. The zero-trace GUE $\mathbf{G}$ is related to a GUE without trace constraint $\mathbf{A}$ by $\mathbf{G} = \mathbf{A} - \mathrm{tr}(\mathbf{A})\mathbf{I}/D$, where $\mathbf{I}$ is an identity matrix. Comparing $\mathbf{G}$ and $\mathbf{A}$, the variance of the diagonal elements of $\mathbf{G}$ is reduced by a factor of $1-1/D$. The arithmetic average of the variance of all the elements of $\mathbf{G}$, which is also proportional to the variance of its eigenvalues, is a factor $1-D^{-2}$ smaller than the variance of all the elements of $\mathbf{A}$. Although this factor of $1-D^{-2}$ was found empirically in [9], it can be derived analytically by comparing $\mathbf{A}$ and $\mathbf{G}$ in $\mathbf{G} = \mathbf{A} - \mathrm{tr}(\mathbf{A})\mathbf{I}/D$. Without the zero-trace constraint, the eigenvalues of a GUE have a joint probability density given by [15, Sec. 3.3]

$$p(\mathbf{x}) = C_{D2}^{-1} \prod_{D \geq i \geq j 1} \left(x_i - x_j\right)^2 \exp\left(-\sum_{i=1}^{D} x_i^2\right) \quad (1)$$

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with $C_{D2} = \pi^{D/2} 2^{-D(D-1)/2} \prod_{n=1}^{D} n!$. The eigenvalues described by (1) correspond to the group delays of a multimode fiber with a normalized variance $\frac{1}{2}D$ without a trace constraint and arbitrary unit. The zero-trace constraint reduces the eigenvalue variance to a slightly smaller value $\frac{1}{2}(D - 1/D)$ [9].

The $D \times D$ Vandermonde determinant gives [15, Sec. 5.3]

$$\det\left[x_i^{j-1}\right]_{i,j=1,2,\ldots,D} = \prod_{D \geq i \geq j \geq 1} \left(x_i - x_j\right) \tag{2}$$

When the probability density (1) is expressed as a determinant, we may obtain [15, Sec. 6.2]

$$\begin{aligned} p(\mathbf{x}) &= \frac{1}{D!} \det\left[\psi_{j-1}(x_i)\right]^2_{i,j=1,2,\ldots,D} \\ &= \frac{1}{D!} \det\left[K_D(x_i, x_j)\right]_{i,j=1,2,\ldots,D} \end{aligned} \tag{3}$$

where $\psi_n(t) = \left(2^n n! \sqrt{\pi}\right)^{-1/2} H_n(t) \exp(-t^2/2)$ is a Hermite function, $H_n(t)$ is a Hermite polynomial, and

$$K_D(x, y) = \sum_{n=0}^{D-1} \psi_n(x) \psi_n(y) \tag{4}$$

is called the correlation function. Comparing the determinants in (2) and (3), the Hermite polynomials provide just the leading terms x_i^{j-1}[15, Sec. 5.3]. According to the properties of determinants, as long as the leading term in each entry is maintained, the entries in the Vandermonde determinant (2) may be replaced by a polynomial without changing the results. In later parts of the calculation, the Hermite function in (3) may be replaced by $\psi_n(t) \propto t^n \exp(-t^2/2)$ without changing the results by merging the exponential of (1) into the determinant (2).

A. Fredholm Determinant

For a finite-dimensional GUE, the delay spread may be studied using the Fredholm determinant [15, Sec. 20.1], which was first derived to study integral equations. The solution of an integral equation gives numerical values of the Fredholm determinant [16], [17]. The joint probability for the maximum and minimum eigenvalues $\Pr(\lambda_{\max} \leq x, \lambda_{\min} \geq y)$ is given by [18], [15, Secs. 6.1.2, 20.1]

$$F^{(D)}(x, y) = E_2(0, J) = \det\left[1 - K_D(x, y)|_J\right], \tag{5}$$

where $J = (-\infty, y) \bigcup (x, +\infty)$ specifies a region, $E_2(0, J)$ denotes no eigenvalue in the region J, and $\det\left[1 - K_D(x, y)|_J\right]$ is the Fredholm determinant for the kernel $K_D(x, y)$ given by (4). The corresponding integral equation for the Fredholm determinant involves the kernel $K_D(x, y)$ integrated over the complement of J. The function $F^{(D)}(x, y)$ can be found by numerically solving the corresponding integral equation [17], [18].

The delay spread has a cumulative distribution function (CDF) given by

$$\Pr(\lambda_{\max} - \lambda_{\min} \leq x) = -\int_{-\infty}^{+\infty} \left.\frac{\partial F^{(D)}(t, y)}{\partial y}\right|_{t=y+x} dy. \tag{6}$$

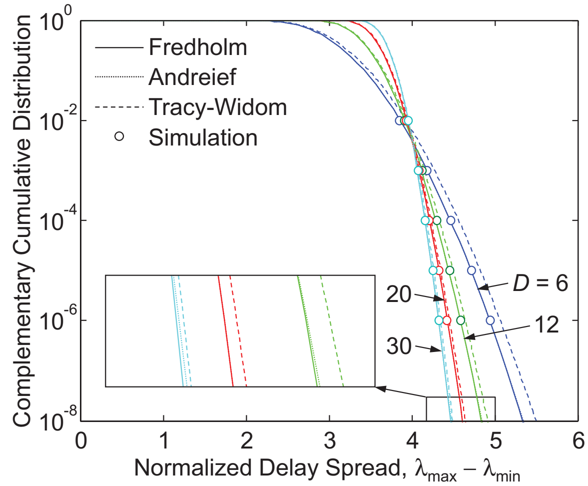

Fig. 1. Complementary cumulative distribution function of the delay spread for multimode fibers with $D = 6$, 12, 20, and 30 modes indicated by different colors. The group-delay variance is normalized to unity. Curves with different line types are evaluated based on the Fredholm determinant (5), the Andréief identity (12), and the Tracy–Widom approximation (16). Simulation results are from [10].

Fig. 1 shows the complementary cumulative distribution function (CCDF) $\Pr(\lambda_{\max} - \lambda_{\min} > x)$ for multimode fibers with $D = 6$, 12, 20, and 30 modes. In Fig. 1, the overall group-delay variance is normalized to unity. Fredholm determinants are calculated using the toolbox from [18].

Numerical calculation of the delay-spread distribution using (6) requires three numerical procedures: first obtaining the Fredholm determinant (5) by solving the corresponding integral equation [17], [18], followed by numerical differentiation $\partial F^{(D)}(x, y)/\partial y$ in the integrand of (6) using the five-point differential formula, and finally the numerical integration (6). The three levels of numerical procedures are tedious and lead to numerical errors at each step. Hence, methods to simply this procedure are highly desirable.

B. Andréief Identity

The joint probability $\Pr(\lambda_{\max} \leq x, \lambda_{\min} \geq y)$ may be evaluated directly by

$$F^{(D)}(x, y) = \int_y^x \cdots \int_y^x p(\mathbf{x}) \mathrm{d}x_1 \cdots \mathrm{d}x_D. \tag{7}$$

The probability density $p(\mathbf{x})$ (1) is the product of two determinants from (3). To simplify later calculations, we use

$$p(\mathbf{x}) = C_{D2}^{-1} \det\left[x_i^{j-1} e^{-x_i^2/2}\right]^2_{i,j=1,2,\ldots D}. \tag{8}$$

The Andréief identity gives:

$$\begin{aligned} &\iiint \det\left[f_i(x_j)\right] \det\left[g_i(x_j)\right] d\mathbf{x} \\ &\qquad = D! \det\left[\int f_i(x) g_j(x) dx\right]. \end{aligned} \tag{9}$$

Given first in [19] and reintroduced in [20], the Andréief identity is the continuous form of the Cauchy–Binet formula [21], also called Gram formula in [15, Sec. A.12].

Using the Andréief identity, we have

$$F^{(D)}(x, y) = \frac{D!}{C_{D2}} \det \left[\int_y^x t^{i+j-2} e^{-t^2} dt \right]_{i,j=1,2,\ldots,D.} \quad (10)$$

Denote the integration in (10) as $I_n(x, y) = \int_y^x t^n e^{-t^2} \mathrm{d}t$. We may obtain iteratively:

$$I_0(x, y) = \frac{\sqrt{\pi}}{2} (\mathrm{erf} x - \mathrm{erf} y)$$
$$I_1(x, y) = \frac{1}{2} \left(e^{-y^2} - e^{-x^2} \right)$$
$$I_n(x, y) = \frac{1}{2} \left(y^{n-2} e^{-y^2} - x^{n-2} e^{-x^2} \right) + \frac{k-2}{2} I_{n-2}(x, y), \quad n \geq 2 \quad (11)$$

We may further obtain

$$-\frac{\partial F^{(D)}(x, y)}{\partial y} = \frac{D!}{C_{D2}} \sum_{n=1}^{D} \det \begin{bmatrix} I_{i+j-2}(x, y), i \neq n \\ y^{i+j-2} e^{-y^2}, i = n \end{bmatrix}_{i,j=1,2,\ldots,D} \quad (12)$$

which may be obtained directly using (11).

Substituting (12) in (6), the delay-spread distribution can be obtained by only one level of numerical integration. Although derived using different methods and for different applications, the distribution (12) is similar to the conditional number distribution for Wishart matrix in [22]. Fig. 1 shows the CCDF $\Pr(\lambda_{\max} - \lambda_{\min} > x)$ for multimode fibers with $D = 6$, 12, 20, and 30 modes calculated by (12). Values calculated using the Andréief identity differ slightly from those computed using the Fredholm determinant. The small difference arises presumably from the multiple levels of numerical calculation involved in using the Fredholm determinant.

C. Approximation Based on Tracy–Widom Distribution

It may be of interest to approximate the distribution of the delay spread for many-mode fibers. For very large matrices, the maximum eigenvalue is described by the celebrated Tracy-Widom distribution [23], [24]. For the GUE described by the distribution (1), the largest eigenvalue is approximately $\sqrt{2D}$ [15, Sec. 6.2]. The Tracy-Widom distribution for a GUE is given by

$$\lim_{D\to\infty} \Pr\left(\frac{\lambda_{\max} - \sqrt{2D}}{2^{-1/2} D^{-1/6}} \leq x \right) = F_2(x), \quad (13)$$

where $\lambda_{\max}$ denotes the maximum eigenvalue and $F_2(x)$ is the CDF for the Tracy–Widom distribution with numerical values calculated in [18], [25].

For many-mode fibers, as shown in [26], the maximum and minimum eigenvalues are independent of each other. The delay spread for a large GUE is given by

$$\lim_{D\to\infty} \Pr\left(\frac{\lambda_{\max} - \lambda_{\min} - 2\sqrt{2D}}{2^{-1/2} D^{-1/6}} \leq x \right) = F_{\mathrm{ds}}(x), \quad (14)$$

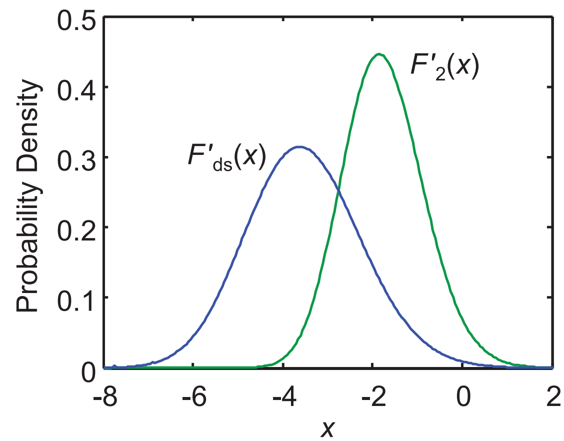

Fig. 2. The Tracy–Widom distribution $F_2'(x)$ and the normalized delay-spread distribution $F_{\mathrm{ds}}'(x)$.

where

$$F_{\mathrm{ds}}(x) = \int_{-\infty}^{+\infty} F_2(x - t) \frac{\mathrm{d}}{\mathrm{d}t} F_2(t) \mathrm{d}t \quad (15)$$

is the CDF of the summation of two independent Tracy–Widom random variables. Fig. 2 shows the Tracy–Widom distribution $F_2'(x)$ and the corresponding delay-spread distribution $F_{\mathrm{ds}}'(x)$ (15). $F_2'(x)$ is calculated using the toolbox from [18].

Fig. 1 also shows the approximate CCDF

$$\Pr(\lambda_{\max} - \lambda_{\min} > x) \approx 1 - F_{\mathrm{ds}} \left[D^{2/3} \left(x\sqrt{1 - D^{-2}} - 4 \right) \right] \quad (16)$$

for fibers with $D = 6$, 12, 20, and 30 modes with group delay variance normalized to unity. The approximation (16) follows directly from (14) for all values of D and takes into account that the group-delay variance for the statistics corresponding to (14) is $\frac{1}{2}(D - 1/D)$. The approximation (16) always overestimates the CCDF, but becomes increasingly accurate as D increases.

III. Discussion

The CCDF of the delay spread governs the temporal memory length required in a digital equalizer for compensating modal dispersion and separating mode-division-multiplexed signals [10]. Following [10], given a value of D and a probability p, we define $u_D(p)$ to be the value of x such that $\Pr(\lambda_{\max} - \lambda_{\min} > x) = p$. Defining σ_{gd} to be the standard deviation (STD) of the group delay, an equalizer length equal to $u_D(p)\sigma_{\mathrm{gd}}$ is sufficient to span the channel memory with probability $1 - p$. For the values of D shown in Fig. 1 and for p of order 10^{-4} to 10^{-6}, $u_D(p)$ is of order 4 to 5.

Fig. 1 shows numerically simulated values of the CCDF of the delay spread taken from [10], which show no observable difference from calculations using the Fredholm determinant or Andréief identity (these two are virtually indistinguishable, as noted previously). In [10], the CCDF is simulated by collecting the difference between the maximum and minimum eigenvalues of many randomly generated zero-trace Hermitian Gaussian random matrices.

The approximation based on the sum of two independent Tracy–Widom random variables always over-estimates the CCDF, but is sufficiently accurate for most engineering purposes for $D \geq 12$. Expression (16) shows that as $D \to \infty$, the delay spread is upper-bounded by four times the STD of group delay. In this limit, the group delay follows a Wigner semicircle distribution [9], which has a finite support equal to four times the STD.

The Tracy–Widom distribution has a mean of -1.77, and the normalized delay spread has a mean of approximately

$$\frac{4 - 3.54D^{-2/3}}{\sqrt{1 - D^{-2}}}. \tag{17}$$

The mean delay spread (17) approaches 4 as D increases. The Tracy–Widom distribution has an STD of 0.902 and the normalized delay spread has an STD of approximately

$$\frac{1.28D^{-2/3}}{\sqrt{1 - D^{-2}}}. \tag{18}$$

The delay-spread STD (18) decreases to zero as $D^{-2/3}$ as D increases. In [14], the delay spread STD decreases asymptotically as $1/D$, slightly faster than (18). The mean delay spread (17) was not studied in [14].

IV. Conclusion

In the strong mode coupling regime, the group delay of multimode fiber has the same statistics as a zero-trace GUE. The delay spread is the difference between the maximum and minimum eigenvalues of a GUE, and its distribution has been determined using three methods. Calculations using the Fredholm determinant are consistent with those using the Andréief identity. An approximation based on the Tracy-Widom distribution becomes accurate when the number of modes D is at least 12.

References

[1] H. R. Stuart, "Dispersive multiplexing in multimode optical fiber," *Science*, vol. 289, no. 5477, pp. 281–283, 2000.

[2] R. Ryf, *et al.*, "Mode-division multiplexing over 96 km of few-mode fiber using coherent 6 × 6 MIMO processing," *J. Lightw. Technol.*, vol. 30, no. 4, pp. 521–531, Feb. 15, 2012.

[3] A. Al Amin, A. Li, S. Chen, X. Chen, G. Gao, and W. Shieh, "Dual-LP_{11} mode 4×4 MIMO-OFDM transmission over a two-mode fiber," *Opt. Express*, vol. 19, no. 17, pp. 16672–16678, 2011.

[4] F. Yaman, N. Bai, B. Zhu, T. Wang, and G. Li, "Long distance transmission in few-mode fibers," *Opt. Express*, vol. 18, no. 12, pp. 13250–13257, 2010.

[5] C. Koebele, *et al.*, "Two mode transmission at 2×100 Gb/s, over 40 km-long prototype few-mode fiber, using LCOS based mode multiplexer and demultiplexer," *Opt. Express*, vol. 19, no. 17, pp. 16593–16600, 2011.

[6] D. Gloge, "Optical power flow in multimode fibers," *Bell Syst. Tech. J.*, vol. 51, no. 8, pp. 1767–1780, 1972.

[7] R. Olshansky, "Mode-coupling effects in graded-index optical fibers," *Appl. Opt.*, vol. 14, no. 4, pp. 935–945, 1975.

[8] M. B. Shemirani, W. Mao, R. A. Panicker, and J. M. Kahn, "Principal modes in graded-index multimode fiber in presence of spatial- and polarization-mode coupling," *J. Lightw. Technol.*, vol. 27, no. 10, pp. 1248–1261, May 15, 2009.

[9] K.-P. Ho and J. M. Kahn, "Statistics of group delays in multimode fiber with strong mode coupling," *J. Lightw. Technol.*, vol. 29, no. 21, pp. 3119–3128, Nov. 1, 2011.

[10] S. Ö. Arik, D. Askarov, and J. M. Kahn, "Effect of mode coupling on signal processing complexity in mode-division multiplexing," *J. Lightw. Technol.*, to be published.

[11] K.-P. Ho and J. M. Kahn, "Mode-dependent loss and gain: Statistics and effect on mode-division multiplexing," *Opt. Express*, vol. 19, no. 17, pp. 16612–16635, 2011.

[12] P. J. Winzer and G. J. Foschini, "MIMO capacities and outage probabilities in spatially multiplexed optical transport systems," *Opt. Express*, vol. 19, no. 17, pp. 16680–16696, 2011.

[13] K.-P. Ho and J. M. Kahn, "Frequency diversity in mode-division multiplexing systems," *J. Lightw. Technol.*, vol. 19, no. 24, pp. 3719–3726, Dec. 15, 2011.

[14] C. Antonelli, A. Mecozzi, M. Shtaif, and P. J. Winzer, "Stokes-space analysis of modal dispersion in fibers with multiple mode transmission," *Opt. Express*, vol. 20, no. 11, pp. 11718–11733, 2012.

[15] M. L. Mehta, *Random Matrices*, 3rd ed. New York: Elsevier, 2004.

[16] C. A. Tracy and H. Widom, "Fredholm determinants, differential equations and matrix models," *Commun. Math. Phys.*, vol. 163, no. 1, pp. 33–72, 1994.

[17] F. Bornemann, "On the numerical evaluation of Fredholm determinants," *Math. Comput.*, vol. 79, no. 270, pp. 871–915, 2010.

[18] F. Bornemann, "On the numerical evaluation of distribution in random matrix theory: A review," *Markov Process. Relat. Fields*, vol. 16, no. 4, pp. 803–866, 2010.

[19] P. M. C. Andréief, "Note sur une relation entre les integrals définies des produits des functions," *Mém. Soc. Sci. Phys. Nat. Bordeaux*, vol. 2, no. 3, pp. 1–14, 1886.

[20] N. G. de Bruijn, "On some multiple integrals involving determinants," *J. Indian Math. Soc.*, vol. 19, pp. 133–151, Apr. 1955.

[21] D. Serre, *Matrix: Theory and Applications*. New York: Springer-Verlag, 2010, sec. 3.2.2.

[22] M. Matthaiou, M. R. McKay, P. J. Smith, and J. A. Nossek, "On the condition number distribution of complex Wishart matrices," *IEEE Trans. Commun.*, vol. 58, no. 6, pp. 1705–1717, Jun. 2010.

[23] C. A. Tracy and H. Widom, "Level-spacing distributions and the Airy kernel," *Phys. Lett. B*, vol. 305, nos. 1–2, pp. 115–118, 1993.

[24] C. A. Tracy and H. Widom, "Level-spacing distributions and the Airy kernel," *Commun. Math. Phys.*, vol. 159, no. 1, pp. 151–174, 1994.

[25] M. Dieng, "Distribution functions for edge eigenvalues in orthogonal and symplectic ensembles: Painlevé representations," *Int. Math. Res. Not.*, vol. 2005, no. 37, pp. 2263–2287, 2005.

[26] F. Bornemann, "Asymptotic independence of the extreme eigenvalues of Gaussian unitary ensemble," *J. Math. Phys.*, vol. 51, no. 2, pp. 023514-1–023514-8, 2010.