The Effect of Neighboring Conductors on the Currents and Fields in Plane Parallel Transmission Lines

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Abstract—In this paper the current distribution is calculated for a microstrip line in the presence of a neighboring strip. The electric field is calculated and the characteristic impedance of the slotted microstrip line is determined. A graph of characteristic impedance is given for odd and even excitations. The calculations are carried out by setting up a singular integral equation which is solved using a finite integral transform. This method has the advantage that the calculations can be generalized in a straightforward manner for the multislotted line.

Introduction

IN THE LAST decade the memory used in large scale digital computers has undergone a rapid development. One of the important trends was to reduce the time necessary to switch the state of the storage element. In recent memories the time necessary to switch one element is on the order of 1 ns while the memory speed is about 100 ns [1].

Therefore the design of a memory array [2] is based on microwave considerations. The array consists of microstrip lines which are usually slotted in order to reduce eddy current losses. The presence of the neighboring conductor influences the current and the field distribution, and therefore this effect has to be accounted for. Coupling effects are often very considerable making the quantitative behavior of the microstrip backward coupler of considerable interest.

The aforementioned considerations also enter in designing integrated circuits for various purposes. For this case and also for the memory array, the actual microstrip contains a layered material between the upper microstrip and the lower groundplate, but the forward coupling of the layer is not considered in the following calculations.

The properties of a single microstrip are well known from calculations done in the early fifties [3]. Later the symmetrical stripline came into prominence [4], partly because it possessed advantageous properties, such as small stray coupling and low loss, and also because the theorists recognized that it was easier to analyze the symmetrical stripline than the microstrip.

In addition to analyzing the single symmetrical stripline in the fundamental mode [4], the cutoff wavelength of the first higher mode has also been calculated [5] and approximate equivalent circuits for the discontinuities have been derived by Oliner [5]. The coupling effect in symmetrical striplines has been analyzed by Cohn, Jones, and Bolljahn [6].

More recently Wheeler [7] obtained explicit solutions for wide microstrips in terms of simple functions. Other recent work [8] shows a continued interest in the properties of striplines.

From the point of view of integrated circuits and memory arrays, a microstrip configuration is much better suited than a symmetrical stripline. However, while the properties of one microstrip line were well known, the fields due to more than one strip and the coupling between the microstrips had not been calculated. One reason for this could be attributed to the fact that the Schwarz–Christoffel integral (successfully applied by Black and Higgins [3d] to one microstrip) leads to a complicated integral when applied to a number of microstrip segments and cannot be evaluated in closed form. Therefore, it was decided to use an integral equation approach. This approach is seldom used [9] due to difficulties in obtaining a solution to the integral equation. Clearly, if the potential theoretic problem is formulated in terms of the *field* produced by an unknown current distribution, then an integral equation with a Hilbert kernel is obtained. This type of integral equation has been treated in the literature [10]. It has been used by Palócz [11] in investigating the field due to a double strip, when even symmetry is present in the field. Generalization to the case of more than two segments have been given by Nickel [12], Palócz [11] and Lewin [13].

In this paper, a finite Hilbert transform [14] is used to calculate the current distribution of coupled elements in a memory array or integrated circuit. The characteristic impedances are calculated and presented in a graph for odd and even symmetry. However, the method itself is more general since finite integral transforms [15] are directly applicable to a large class of problems in electrophysics.

Formulation of the Integral Equation

Consider the slotted microstrip line whose cross section is shown in Fig. 1. Note that its dimensions have been normalized so that the total width of the configuration is two. Normalized linear dimensions are obtained by dividing all actual lengths by one half the total actual width. A stripline thus normalized will have the same characteristics as the line of Fig. 1 insofar as the quantities of interest in this paper are concerned.

The ground plane is replaced by image electrodes as shown in Fig. 2. As usual, the images carry the same currents as the actual electrodes except the sign of the current is

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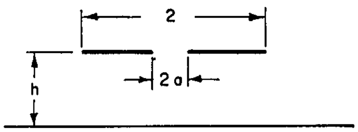

Fig. 1. The stripline configuration of interest using normalized dimensions. The ground plane is of infinite extent.

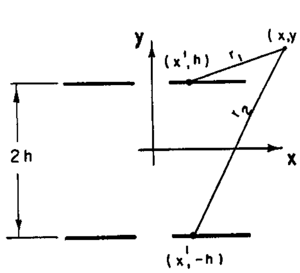

Fig. 2. The equivalent structure obtained by the method of images, showing the coordinate system used.

reversed (i.e., $i(x, h) = -i(x, -h)$). This substitution will aid in the formulation of the integral equation. The following assumptions are also made for that purpose.

1) The configuration is plane parallel and infinite in the z direction (the z direction is perpendicular to the cross section of Fig. 1).
2) The system is lossless and located in free space.
3) The ground plane is infinite in extent.
4) The thickness of each strip is zero.
5) Only a transverse electromagnetic (TEM) wave exists, and it propagates in the z direction.
6) The excitation is harmonic.

By assumptions 5 and 6 all fields vary as $\exp[j(\omega t - kz)]$, where k is the free space wavenumber equal to $\omega(\epsilon_0\mu_0)^{1/2}$. For example the scalar potential

$$\Psi(x, y, z, t) = \psi(x, y) \exp[j(\omega t - kz)]. \tag{1}$$

Since $\exp[j(\omega t - kz)]$ is a common factor of all field equations it will be suppressed. Under these assumptions, it follows [11] that

$$\psi(x, y) = \frac{1}{2\pi}\left(\frac{\mu_0}{\epsilon_0}\right)^{1/2} \iint_{(S)} i(x', y') \ln(r/R_0) dS', \tag{2}$$

where $\overline{R}_0$ is a vector from the source point (x', y') to an arbitrary reference point (the zero of potential), and $\bar{r}$ is a vector from the source point to the field point (x, y). The origin will be taken as the zero of potential. Referring to Fig. 2, it is seen that $i(x', y')$ is nonzero only for $a \leq |x'| \leq 1$ and $y' = \pm h$. That is $i(x', y')$ is of the form $i(x')[\delta(y'-h) - \delta(y'+h)]$. Since the only y' dependence is in the delta functions, integrating (2) with respect to y' yields

$$\begin{aligned}\psi(x, y) &= \frac{1}{2\pi}\left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}\left[\int_{-1}^{-a} + \int_{a}^{1}\right] i(x') \cdot \left[\ln\left(\frac{r_1}{R_0}\right) - \ln\left(\frac{r_2}{R_0}\right)\right] dx' \\ &= \frac{1}{2\pi}\left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}\left[\int_{-1}^{-a} + \int_{a}^{1}\right] \cdot i(x') [\ln r_1 - \ln r_2] dx'\end{aligned} \tag{3}$$

where, from the diagram,

$$r_{1_2} = [(x - x')^2 + (y \mp h)^2]^{1/2}. \tag{4}$$

To obtain the integral equation for $i(x')$ the electric field will be computed and its tangential component E_x set equal to zero at the electrodes. Since the wave is TEM, the electric field is obtained by taking the negative transverse gradient of Ψ. This gradient operator refers to the (x, y) points; since these are independent of the (x', y') coordinates over which the integral in (3) is taken, the order of operation may be interchanged so that using (3) and (4) we obtain

$$\begin{aligned}\vec{E} = \frac{1}{2\pi}\left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}\left[\int_{-1}^{-a} + \int_{a}^{1}\right]\Bigg\{&(x - x') \cdot \left[\frac{1}{r_2^2} - \frac{1}{r_1^2}\right]\vec{x}_0 \\ &+ \left[\frac{y+h}{r_2^2} - \frac{y-h}{r_1^2}\right]\vec{y}_0\Bigg\} i(x') dx'\end{aligned} \tag{5}$$

where $\bar{x}_0$ and $\bar{y}_0$ are unit vectors in the x and y directions, respectively. Requiring that E_x vanish at $y = \pm h$ for $a \leq |x| \leq 1$ yields

$$0 = \left(\int_{-1}^{-a} + \int_{a}^{1}\right) i(x')(x - x') dx' \left[\frac{1}{(x - x')^2 + (2h)^2} - \frac{1}{(x - x')^2}\right] \tag{6}$$

or

$$\left[\int_{-1}^{-a} + \int_{a}^{1}\right] \frac{i(x') dx'}{x - x'} = \left[\int_{-1}^{-a} + \int_{a}^{1}\right] \frac{i(x')(x - x') dx'}{(x - x')^2 + 4h^2}. \tag{7}$$

The left-hand side of (7) is considered as a Cauchy principal value of the integral. That is, for $a < x < 1$ the integral is taken to mean

$$\lim_{\epsilon \to 0}\left[\int_{-1}^{-a} + \int_{a}^{x-\epsilon} + \int_{x+\epsilon}^{1}\right] \frac{i(x')}{x - x'} dx'. \tag{8}$$

The analogous definition is used for $-1 < x < -a$.

Thus an integral equation (7) defining $i(x)$ has been derived.

Solution of the Integral Equation

This integral equation involves a finite Hilbert transform, so the problem may be considered to be one of finding the inverse transform. It is fortunate that this type of integral equation also arises in aerodynamics and solutions from this discipline can be applied. Palócz [11] discusses this connection in his paper. In particular Tricomi [14c] has shown that if

$$\left[\int_{-1}^{-a} + \int_{a}^{1}\right]\frac{i(x')}{x - x'}dx' = G(x) \tag{9}$$

then

$$i(x) = \left\{\left[C_0 + C_1x - \frac{1}{\pi}\int_{-a}^{a}\frac{\sqrt{a^2 - z^2}\,F(z)dz}{x - z}\right] \cdot \text{sgn}\,(x) - \sqrt{x^2 - a^2}F(x)\right\}\frac{1}{\sqrt{(1 - x^2)(x^2 - a^2)}}, \tag{10}$$

where

$$F(x) = \frac{1}{\pi^2}\left[\int_{-1}^{-a} + \int_{a}^{1}\right]\frac{\sqrt{1 - x'^2}}{x - x'}G(x')dx'. \tag{11}$$

Using (7), $G(x)$ for the problem of interest is given by

$$G(x) = \left[\int_{-1}^{-a} + \int_{a}^{1}\right]\frac{i(x')(x - x')dx'}{(x - x')^2 + 4h^2}\cdot \tag{12}$$

Using these definitions for $F(x)$ and $G(x)$, (10) is the solution (inverse transform) of the integral equation given in (7). The result is considerably simplified if $i(x)$ is either an even or odd function of x. Since (7) is linear in $i(x)$ no generality is lost by solving separately for odd and even current distributions.

We consider the odd excitation first. If the excitation is an odd function of x (that is, $i(-x) = -i(x)$) then C_1 must be equal to zero (see (10)). Any other value of C_1 leads to an even component of current. Hence

$$i(x) = \left[C_0\,\text{sgn}\,(x) - \frac{\text{sgn}\,(x)}{\pi}\int_{-a}^{a}\frac{\sqrt{a^2 - z^2}}{x - z}F(z)dz - \sqrt{x^2 - a^2}F(x)\right]\frac{1}{\sqrt{(1 - x^2)(x^2 - a^2)}}\cdot \tag{13}$$

Evidently this type of current distribution leads to a simplification in the evaluation of $G(x)$ and $F(x)$ as well. Considering the expression for $G(x)$ given in (12), it would facilitate evaluation to express this as one integral from a to 1. This can be done by noting that for $i(x)$ odd,

$$\int_{-1}^{-a}\frac{i(x')(x - x')dx'}{(x - x')^2 + 4h^2} = \int_{a}^{1}\frac{-\,i(x')(x + x')dx'}{(x + x')^2 + 4h^2}, \tag{14}$$

so (12) becomes

$$G(x) = \int_{a}^{1} i(x')\left[\frac{x - x'}{(x - x')^2 + 4h^2} - \frac{x + x'}{(x + x')^2 + 4h^2}\right]dx', \tag{15}$$

or

$$G(x) = 2\int_{a}^{1}\frac{i(x')\,x'\,(x^2 - x'^2 - 4h^2)dx'}{[(x - x')^2 + 4h^2][(x + x')^2 + 4h^2]}\cdot \tag{16}$$

If x is replaced by $-x$ in (16) there is no change in the value of $G(x)$. Hence $G(x)$ is an even function.

With this knowledge $F(x)$ given by (11) may be rewritten as

$$F(x) = \frac{1}{\pi^2}\int_{a}^{1}\sqrt{1 - x'^2}G(x') \cdot \left(\frac{1}{x - x'} + \frac{1}{x + x'}\right)dx' \tag{17}$$

or

$$F(x) = \frac{2x}{\pi^2}\int_{a}^{1}\frac{\sqrt{1 - x'^2}\,G(x')}{(x^2 - x'^2)}dx'. \tag{18}$$

Equation (18) shows explicitly that $F(x)$ is an odd function. However, (17) is a more economical form for computer evaluation. Using the fact that $F(x)$ is odd, the second term in (13) can be simplified to an integral over positive values of z only.

An iterative procedure can now be used to obtain the current distribution for each symmetry (odd and even). For a given value of a, one can obtain the current distribution for infinite h, since then $G(x)$ is everywhere zero. Using (11) it is seen that then $F(x)$ is also everywhere zero. But in that case (10) reduces to

$$i(x) = [C_0 + C_1x]\frac{\text{sgn}\,(x)}{[(1 - x^2)(x^2 - a^2)]^{1/2}} \tag{19}$$

so that in the odd mode, where C_1 is zero, the current is uniquely determined, within a constant factor C_0. C_0 is adjusted so that the total current in one strip is equal to one. Since this is the current for infinite h, it is only a first approximation to the current distribution when h is finite. Thus this current is labeled $i_0(x)$. Using $i_0(x)$ in (15), $G_0(x)$, an approximation to $G(x)$, is computed. (This is done for the particular value of h that is of interest.) Using (17), $F_0(x)$ is then computed and used as an approximation to the true $F(x)$. At this point in the process, (13) can be utilized to obtain a second approximation to $i(x)$ for the given value of a and h. This is labeled $i_1(x)$ and it is used to generate $G_1(x)$ and $F_1(x)$, which can then be used to obtain $i_2(x)$. In a similar manner, $i_3(x)$, $i_4(x)$, etc. may be calculated. In this work, the iterative procedure was terminated when the current changed by less than 2 percent at each point.

It should be noted that the solution for the even mode is carried out in an analogous manner with the exception that then C_0 is zero. Equations analogous to (13), (15), and (17) are used. Not surprisingly, $G(x)$ becomes an odd function and $F(x)$ becomes an even one, the opposite situation from the case where $i(x)$ was odd. A more detailed analysis for the even mode is given by Palócz [11].

The results of this process are shown in Fig. 3 for representative values of the parameters. Note that for both modes the current distribution becomes flatter as h is decreased. However, the even mode and odd modes are almost mirror images, using the middle of the strip as the axis of symmetry.

Characteristic Impedance

Now that the current distribution is known, it is possible to calculate the characteristic impedance Z_0 of the line. Equation (5) gives an expression for the x and y components

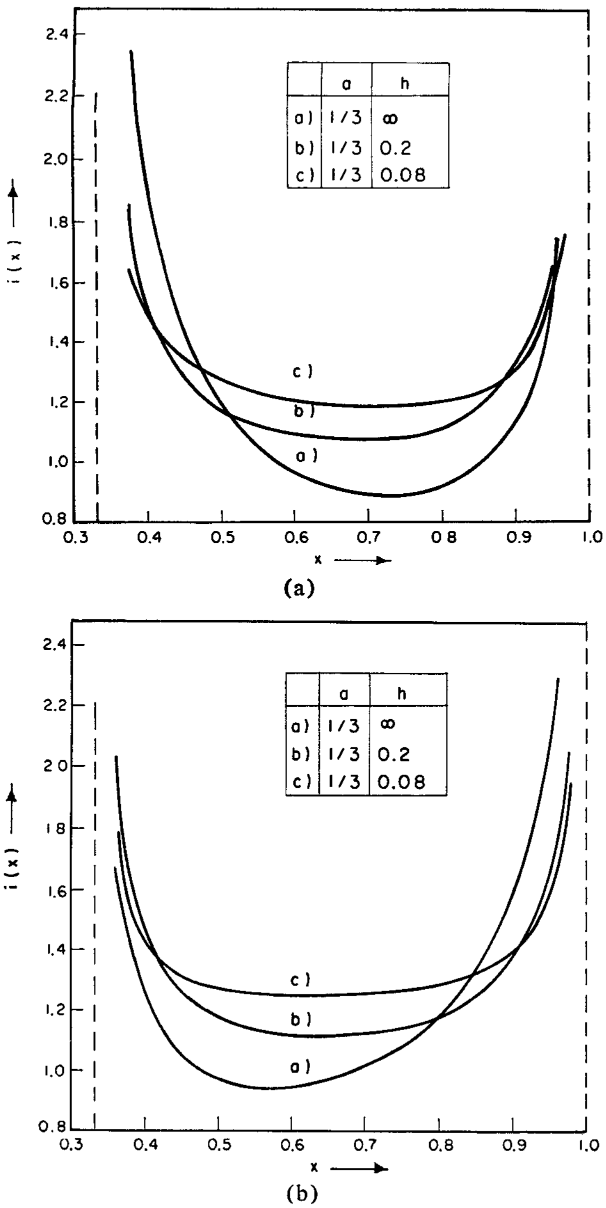

Fig. 3. Effect of strip to ground plane distance on current distribution. Fig. 3(a) refers to odd excitation, Fig. 3(b) to even excitation.

of the electric field (E_x and E_y, respectively) in terms of the current distribution. Thus, E_x and E_y can now be calculated and integrated along the proper path to yield potential difference.

For the odd mode, the transmission line can be excited as shown in Fig. 4(a). If the line is properly terminated or infinite in extent (as assumed), the driving point impedance is equal to the characteristic impedance. Thus,

$$Z_0 = \frac{V}{I_0} \tag{20}$$

where

$$V = \int_{-a}^{+a} [E_x]_{y=h} dx \tag{21}$$

and

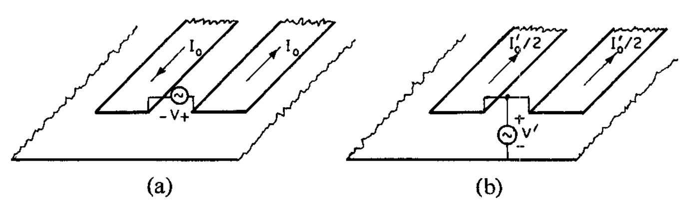

Fig. 4. (a) Excitation of the line in the odd mode. (b) Excitation of the line in the even mode.

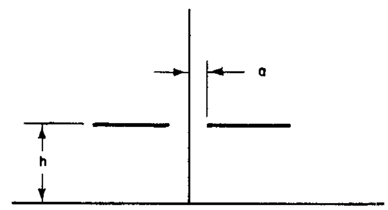

Fig. 5. An additional, half infinite ground plane can be added in the odd mode.

$$I_0 = \int_a^1 i(x) dx, \tag{22}$$

so that V is the voltage between the strips and I_0 is the current in one strip. Suitable approximations for the integrals can be derived (the integrands are finite except at the endpoints) so that Z_0 can be calculated.

It is interesting to note that due to symmetry, the plane $x=O$ is at the same potential as the ground plane. Hence a semi-infinite conducting plane can be added, as shown in Fig. 5, without affecting the properties of the line in the *odd* mode. The characteristic impedance of one strip to ground in the odd mode, Z_{0o} is thus given by

$$Z_{0o} = \frac{\int_0^a E_x dx}{I_0} = \frac{1}{2} Z_0. \tag{23}$$

The even mode can be excited as shown in Fig. 4(b). The driving point impedance (and hence the characteristic impedance) is now

$$Z_0' = \frac{V'}{I_0'} \tag{24}$$

where

$$V' = \left[\int_0^h E_y \right]_{x=x_0} dy, \tag{25}$$

and x_0 is any (fixed) value of x satisfying $a < x_0 < 1$, and

$$I_0' = \left[\int_{-1}^{-a} + \int_a^1 \right] i(x) dx = 2 \int_a^1 i(x) dx. \tag{26}$$

Since each segment carries only half the total current, the characteristic impedance per segment to ground in the even mode Z_{0e} is

$$Z_{0e} = \frac{V'}{I_0'/2} = 2 Z_0'. \tag{27}$$

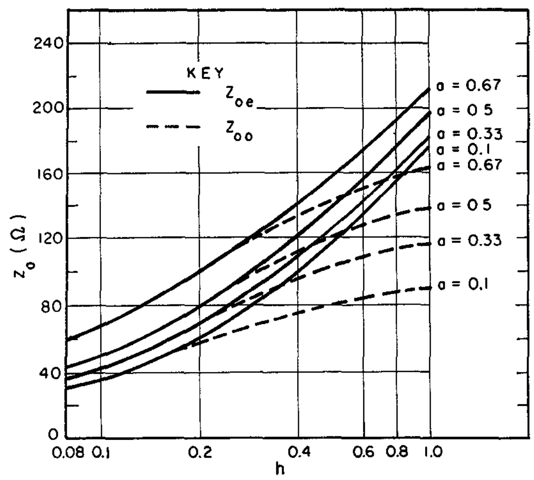

Fig. 6. Characteristic impedance of one segment to ground for the even excitation (Z_{0e}) and for the odd excitation (Z_{0o}). Refer to Fig. 1 for the definition of the distances a and h.

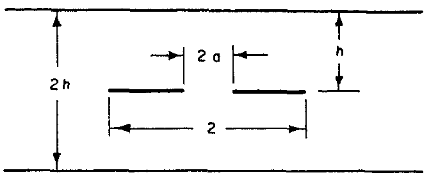

Fig. 7. The symmetrical stripline configuration analyzed by S. B. Cohn [6a].

The plane $x=0$ is no longer at ground potential so it is not possible to add the additional ground plane without affecting the properties of the line.

Values of Z_{0o} and Z_{0e} were calculated for various values of the parameters a and h. The results are depicted in the form of a graph in Fig. 6. The accuracy of Z_0 is related to the accuracy of the current distribution. For large values of h, fewer iterations are necessary to obtain the current distribution, since in this case the initial current distribution is a better approximation to the true distribution (see Fig. 3). Thus, errors due to roundoff and numerical integration are smaller for large values of h, and the accuracy in $i(x)$ and Z_0 is better. For the larger values of h and a, the accuracy in Z_0 is on the order of 3 percent. For small h or a the accuracy is somewhat less, being on the order of 10 percent. For small values of h as many as five iterations were required, while for $h \geq 0.2$ and $a \geq 0.3$ the use of two iterations was found to be sufficient, explaining the difference in accuracy.

The problem in obtaining an accurate result for small values of h is also due to the fact that for h equal to zero the defining integral equation (7) is a trivial identity, and thus cannot be used.

Cohn [6a] analyzed a similar configuration, the so-called symmetrical stripline shown in Fig. 7. The additional ground plane yields a second degree of symmetry which facilitates the computations. In his comparison of Z_{0o} and Z_{0e} for this geometry, Cohn notes that Z_{0o} should be smaller than Z_{0e}, since in this geometry also, an additional ground plane can be added at $x=0$ without affecting the line in the odd mode. The added capacitance from the strip to ground in the odd mode thus lowers the characteristic impedance. This same behavior should occur for the configuration of Fig. 1 and indeed, within the tolerances previously stated, Z_{0o} was smaller than Z_{0e}.

Another trend can be predicted by a similar argument. As the ground plane is removed (h becomes large) Z_{0e} approaches infinity since the capacitance from a strip to ground approaches zero. However, in the odd mode there is the additional capacitance to the additional (imaginary) ground plane. This capacitance does not approach zero, but rather is constant. Thus Z_{0o} should approach a finite limit as h approaches infinity. Fig. 6 demonstrates this behavior, bearing out the prediction.

References

[1a] M. S. Blois, "Preparation of thin magnetic films and their properties," *J. Appl. Phys.*, vol. 26, pp. 975–980, August 1955.
[1b] A. Yelon, O. Voegeli, and E. W. Pugh, "Switching properties of single crystal Ni–Fe films," *J. Appl. Phys.*, vol. 36, pp. 101–108, January 1965.
[1c] W. Anacker, G. F. Bland, P. Pleshko, and P. F. Stuckert, "On the design and performance of a small 60 ns destructive readout magnetic film memory," *IBM J. Res. Develop.*, vol. 10, pp. 41–50, 1966.
[2] E. W. Pugh, V. T. Shahan, and W. T. Siegle, "Device and array design for a 120 ns magnetic film main memory," *IBM J. Res. & Develop.*, vol. 11, pp. 169–178, 1967.
[3a] D. D. Grieg and H. F. Engelmann, "Microstrip—A new transmission technique for the kilomegacycle range," *Proc. IRE*, vol. 40, pp. 1644–1650, December 1952.
[3b] F. Assadourian and E. Rimai, "Simplified theory of microstrip transmission systems," *Proc. IRE*, vol. 40, pp. 1651–1657, December 1952.
[3c] E. G. Fubini, W. E. Fromm, and H. S. Keen, "New techniques for high-Q strip microwave components," *1954 IRE Convention Rec.*, p. 8, pp. 91–97; "Microwave application of high-Q strip components," *ibid.*, pp. 98–103.
[3d] K. G. Black and T. J. Higgins, "Rigorous determination of the parameters of microstrip transmission lines," *IRE Trans. Microwave Theory and Techniques*, vol. MTT-3, pp. 93–113, March 1955.
[4] See the special issue of the *IRE Trans. Microwave Theory and Techniques*, vol. MTT-3, March 1955.
[5] A. A. Oliner, "Theoretical developments in symmetrical strip transmission lines," *Proc. Symp. on Modern Advances in Microwave Techniques*, (November 8–10, 1954), pp. 379–402.
[6a] S. B. Cohn, "Shielded coupled-strip transmission line," *IRE Trans. Microwave Theory and Techniques*, vol. MTT-3, p. 29–38, October 1955.
[6b] ——, "Characteristic impedance of the shielded strip transmission line," *IRE Trans. Microwave Theory and Techniques*, vol. MTT-2, pp. 52–57, July 1954.
[6c] ——, "Problems in strip transmission lines," *IRE Trans. Microwave Theory and Techniques*, vol. MTT-3, pp. 119–126, March 1955.
[6d] E. M. T. Jones and J. T. Bolljahn, "Coupled-strip-transmission-line filters and directional couplers," *IRE Trans. Microwave Theory and Techniques*, vol. MTT-4, pp. 75–81, April 1956.
[7] H. A. Wheeler, "Transmission-line properties of parallel wide strips by a conformal-mapping approximation," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-12, pp. 280–289, May 1964.
[8a] J. W. Duncan, "Characteristic impedances of multiconductor strip transmission lines," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-13, pp. 107–118, January 1965.
[8b] H. A. Wheeler, "Transmission-line properties of parallel strips separated by a dielectric sheet," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-13, pp. 172–185, March 1965.
[8c] R. W. Burton and R. W. P. King, "An experimental investigation

of a two-slot transmission line on non-planar surfaces," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-13, pp. 303–306, May 1965.

[8d] J. P. Shelton, Jr., "Impedances of offset parallel-coupled strip transmission lines," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-14, pp. 7–15, January 1966.

[8e] S. Yamamoto, T. Azakami, and K. Itakura, "Coupled strip transmission line with three center conductors," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-14, pp. 446–461, October 1966.

[8f] ——, "Slit-coupled strip transmission lines," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-14, pp. 542–553, November 1966.

[8g] H. C. Okean, "Properties of a TEM transmission line used in a microwave integrated circuit application," *IEEE Trans. on Microwave Theory and Techniques*, (Correspondence), vol. MTT-15, pp. 327–328, May 1967.

[8h] H. Guckel, "Characteristic impedances of generalized rectangular transmission lines," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-13, pp. 270–274, May 1965.

[8i] M. K. Krage and G. I. Haddad, "The characteristic impedance and coupling coefficient of coupled rectangular strips in a waveguide," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-16, pp. 302–307, May 1968.

[9] G. Wendt, "Statische Felder und Stattionäre Ströme," *Encyclopedia of Physics*, S. Flugge, Ed., vol. 16. Berlin: Springer, 1958 (see especially page 37).

[10a] N. I. Muskhelisvili, *Singular Integral Equations.* Groningen, Netherlands: Noordhoff, 1967.

[10b] F. G. Tricomi, *Integral Equations.* New York: Interscience Publishers, 1957.

[10c] N. P. Vekua, *Systems of Singular Integral Equations.* Groningen, Netherlands: Noordhoff, 1967.

[11] I. Palócz, "The integral equation approach to currents and fields in plane parallel transmission lines," *J. Math. Mech.*, vol. 14, pp. 541–560, April 1966.

[12] K. Nickel, "Losung eines Integralgleichungssystems aus der Tragflügeltheorie," *Math. Z.*, vol. 54, pp. 81–96, 1951.

[13] L. Lewin, "The solution of singular integral equations over a multiple interval and applications to multiple diaphragms in rectangular waveguides," *SIAM J. of Appl. Math.*, vol. 16, pp. 417–438, March 1968.

[14a] F. G. Tricomi, "On the finite Hilbert transform," *Quart. J. of Math.*, vol. 2, pp. 199–211, 1951.

[14b] H. Sohngen, "Zur Theorie der Endlichen Hilbert Transformation," *Math. Z.*, vol. 60, pp. 31–51, 1954.

[14c] F. G. Tricomi, "The airfoil equation for a double interval," *Z. Angew. Math. Phys.*, vol. 2, pp. 402–406, 1951.

[15a] J. C. Tranter, *Integral Transforms in Mathematical Physics.* New York: Wiley, 1951.

[15b] R. V. Churchill, "The operational calculus of Legendre transforms," *J. Math. and Phys.*, vol. 33, pp. 165–177, 1955.

[15c] G. Cinelli, "An extension of the finite Hankel transform and applications," *Internat. J. Engineering Science*, vol. 3, pp. 539–559, October 1965.

[15d] D. Naylor, "On a finite Lebedev transform," *J. Math. Mech.*, vol. 12, pp. 375–383, 1963.

From Approximations to Exact Relations for Characteristic Impedances

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Abstract—**Approximations for the characteristic impedance of a special two conductor stripline and for the general function K/K' are derived by a transformation method recently described in the literature; the first, second, and third approximations having a greatest relative error of the order of 10^{-3}, 10^{-6}, and 10^{-12}, respectively. They can be introduced into an algorithm which is based on elementary conformal mappings, and thus further approximations with rapidly vanishing errors can be derived. The results agree with those for the function K/K' obtained by elliptic integral theory. Obviously no such theory is needed to calculate characteristic impedances or the function K/K' with arbitrary accuracy. The advantages of the new method are illustrated for shielded coupled-strip transmission lines, for which an extended diagram with extreme parameter values has been worked out.**

I. Introduction

A METHOD has been developed [1] to achieve good approximations for the characteristic impedances of some parallel and conical transmission lines, and to improve these approximations step by step, so that at last arbitrarily good approximations, i.e., exact values, can be obtained. The interesting fact is that up to now these characteristic impedances could be given exactly only by means of elliptic integrals, but that in [1] the theory of elliptic integrals was not needed at all. The calculation was made by various conformal mappings. In the following paper [2] a connection was found between the theory in [1] and the theory of transformations of elliptic integrals, in particular the Landen transform. From this connection a general and very advantageous algorithm was derived to get the function K/K'. In the first part of this paper it will be proved that it is possible to get the useful and general formulas of [2] without going back to elliptic integral theory. Elementary conformal mappings as in [1] will suffice.

In the second part, two coupled strip transmission lines are treated, as an interesting technical and somewhat more complicated example. Thus it will be illustrated that the new method of calculation is simpler than the conventional one, that it is generally applicable, and that the results agree with those obtained by the conventional theory. Further, it is shown that even in cases with extreme design parameters, needed, for example, for new developments, the new method gives the results faster and more exactly by elementary formulas than they can be obtained by tabulated elliptic integral functions.

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