Analytical modeling of matching pursuit

Shahram Shirani, Alireza Shoa
Department of Electrical and Computer Engineering
McMaster University

Outline

- Introduction to matching pursuit
- Problem of optimum dictionary size
- Modeling approach
- Verification of the model
- Quantization
- Validation of the model
- Future work

Introduction to matching pursuit

- Matching pursuit is a greedy algorithm that decomposes a signal f into bases selected from an overcomplete dictionary of bases.

$$|\langle R^i f, g_{\gamma_i} \rangle| = \sup_{g_\gamma \in D} |\langle R^i f, g_\gamma \rangle| \tag{1}$$

  where D is the set of all dictionary vectors.

- The residual signal is projected on g_{γ_i}

$$R^i f = \langle R^i f, g_{\gamma_i} \rangle g_{\gamma_i} + R^{i+1} f \tag{2}$$

  or

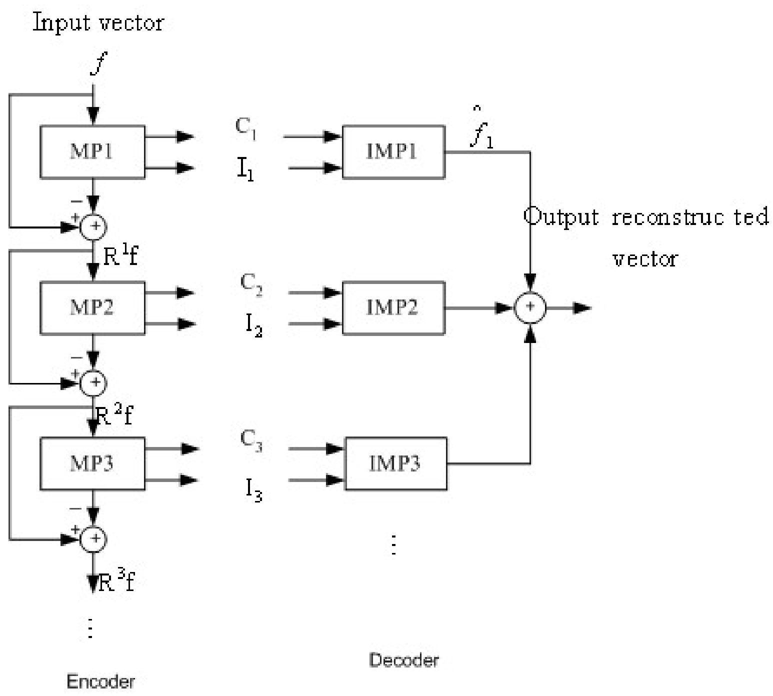
$$R^i f = c_i g_{\gamma_i} + R^{i+1} f \tag{3}$$

  where $c_i \triangleq \langle R^i f, g_{\gamma_i} \rangle$.

- At the first stage $R^0 f$ is the signal f.

$$f = \sum_{i=0}^{k-1} \langle R^i f, g_{\gamma_i} \rangle g_{\gamma_i} + R^k f \tag{4}$$

Introduction to matching pursuit



Introduction to matching pursuit

- g_{γ_i} and $R^{i+1}f$ are orthogonal:

$$\|R^i f\|^2 = |\langle R^i f, g_{\gamma_i}\rangle|^2 + \|R^{i+1} f\|^2 \tag{5}$$

- The signal energy decomposition:

$$\|f\|^2 = \sum_{i=0}^{k-1} |\langle R^i f, g_{\gamma_i}\rangle|^2 + \|R^k f\|^2 \tag{6}$$

- It has been shown that:

$$\|R^i f\| \leq 2^{-\lambda i}\|f\| \tag{7}$$

- Therefore $\|R^i f\|$ converges to zero as i increases.

Dictionary Design

- Dictionary Structure: depends on the distribution of the signal.
  - Gabor dictionaries are the most widely used
- Size of the dictionary:
  - *Large:* more compactness; more bits to address each dictionary vector
  - *Small:* less compactness; fewer bits to address each dictionary vector.
- An analytical method is proposed to find the relationship between the operational bit rate and the dictionary size.
- This relationship can be used to find the optimum dictionary size.

Assumptions

- Suppose:
  - the dictionary has M elements.
  - the dimension of the signal is N.
  - the number of quantization levels is q.
- The matching pursuit iterations must continue until the distortion is less than E^2.
- The normalized signal is assumed to have a uniform distribution.

Distortion criterion

- The matching pursuit iterations must continue until:

$$D_{MP} \leq E^2 \tag{8}$$

- Let

$$\vec{r_i} \triangleq \frac{R^i f}{\|R^{i-1} f\|} \tag{9}$$

and

$$r_i \triangleq \|\vec{r_i}\| \tag{10}$$

Then:

$$\|R^i f\| = \|R^{i-1} f\|.r_i \tag{11}$$

Distortion criterion

- Substituting $R^0 f$ by f :

$$\|R^k f\| = \|f\| r_1 r_2 \ldots r_k \tag{12}$$

- We had:

$$D_{MP} = \|R^k f\|^2 \tag{13}$$

- Therefore:

$$(\|f\| r_1 r_2 \ldots r_k)^2 \leq E^2 \tag{14}$$

  or

$$\|f\| r_1 r_2 \ldots r_k \leq E \tag{15}$$

Matching pursuit rate

- The rate can be written as:

$$\text{Rate} = k\frac{\log_2 M + \log_2 q}{N} \tag{16}$$

  where k is the number of stages of matching pursuit.

- The average rate is

$$E(\text{Rate}) = E\{k\frac{\log_2 M + \log_2 q}{N}\} = \frac{\log_2 M + \log_2 q}{N}E(k) \tag{17}$$

- The probability that exactly k stages are required to meet the energy constraint:

$$p(\mathbf{k} = k) = p(r_1 r_2..r_k \leq E/\|f\|) - p(r_1 r_2...r_{k-1} \leq E/\|f\|) \tag{18}$$

Optimum Dictionary for Uniform Signals

- The normalized signal $f/||f||$ is a vector on the surface of a unit N-sphere.
- The dictionary vectors are on the surface of a unit N-sphere.
- For any dictionary vector we include the negative of that vector in the dictionary. Therefore, the size of the dictionary is now $2M$.
- This changes finding the maximum inner product to finding the minimum distance between $f/||f||$ and the dictionary vectors.
- The signal is uniformly distributed on the surface of the N-sphere therefore in the optimum dictionary
  - The Voronoi regions are identical.
  - The shape of the Voronoi regions must approach the shape of a spherical cap.

Geometric modelling

- The surface area of each Voronoi region is:

$$S = \frac{A_N}{2M} \tag{19}$$

where A_N is the surface area of an N-sphere.

- The volume and the surface area of an N-sphere are:

$$V_N = \frac{\pi^{N/2}}{(N/2)!} r^N = V(N) r^N \tag{20}$$

$$A_N = N \frac{\pi^{N/2}}{(N/2)!} r^{N-1} = A(N) r^N \tag{21}$$

- Therefore:

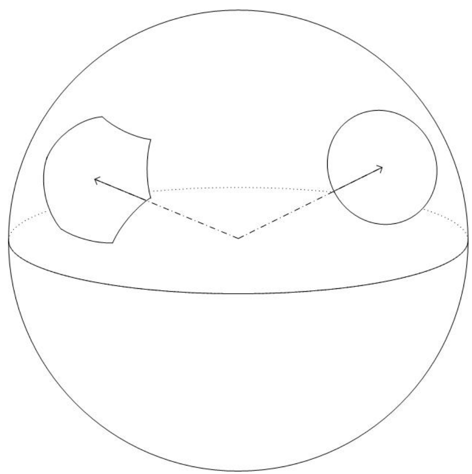
$$S = \frac{A(N)}{2M} \tag{22}$$

Modelling the Voronoi regions by spheres

- The Voronoi regions can be approximated by $(N-1)$-spheres with the same volume as S.

$$V_{N-1} = S \quad \text{or} \quad V(N-1)\mathcal{R}^{N-1} = \frac{A(N)}{2M} \tag{23}$$

$$\mathcal{R} = \left(\frac{A(N)}{2M.V(N-1)}\right)^{\frac{1}{N-1}} = t(N) \times M^{-\frac{1}{N-1}} \tag{24}$$

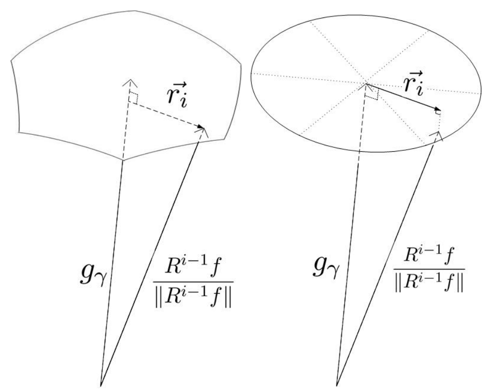


Distribution of the residual signal

Dividing both sides of $R^{i-1}f = \langle R^{i-1}f, g_\gamma \rangle g_\gamma + R^i f$ by $\|R^{i-1}f\|$:

$$\frac{R^{i-1}f}{\|R^{i-1}f\|} = \langle \frac{R^{i-1}f}{\|R^{i-1}f\|}, g_\gamma \rangle g_\gamma + \vec{r_i} \tag{25}$$

Since the normalized signal is uniformly distributed on the surface of the unit sphere, $\vec{r_i}$ is uniformly distributed in the volume of the $N-1$-sphere with radius $\mathcal{R}$



PDF of the residue

- r_i random variable corresponding to $r_i = ||\vec{r_i}||$ can be found by:

$$F_{r_i}(r) = P(r_i \leq r) = \frac{V(N-1) \times r^{N-1}}{V(N-1)\mathcal{R}^{N-1}} = \frac{r^{N-1}}{\mathcal{R}^{N-1}} \tag{26}$$

Therefore:

$$f_{r_i}(r) = \frac{dF_{r_i}(r)}{dr} = \frac{(N-1) \times r^{N-2}}{\mathcal{R}^{N-1}} \quad 0 \leq r \leq \mathcal{R} \tag{27}$$

Finding the number of stages required

- We showed that the distortion is:

$$D_{MP} = R^k f = \|f\| r_1 r_2 ... r_k \tag{28}$$

- Therefore the distortion criterion can be written as:

$$r_1 r_2 ... r_k \leq \zeta \tag{29}$$

where

$$\zeta = \frac{E}{\|f\|} \tag{30}$$

- Let r_i' be defined as:

$$r_i' = \frac{r_i}{\mathcal{R}} \tag{31}$$

- The PDF of r_i' can be found by:

$$f_{r_i'}(r') = (N-1) r'^{N-2} \quad 0 \leq r' \leq 1 \tag{32}$$

Finding the number of stages required

- Equation (32) can be written as:

$$f_{r'}(r') = \alpha r'^{\beta} \quad 0 \le r' \le 1 \tag{33}$$

where:

$$\alpha = N - 1 \quad \text{and} \quad \beta = N - 2 \tag{34}$$

- The distortion criterion can be written as:

$$\mathcal{R}^k r'_1 r'_2 ... r'_k \le \zeta \tag{35}$$

- Substituting equation (35) into equation (18), the probability that exactly k stages are needed to satisfy the distortion constraint will be:

$$p(\mathbf{k} = k) = p(r'_1 r'_2 .. r'_k \le \frac{\zeta}{\mathcal{R}^k}) - p(r'_1 r'_2 ... r'_{k-1} \le \frac{\zeta}{\mathcal{R}^{k-1}}) \tag{36}$$

Lemma

- **Lemma 1** *Suppose* x *and* y *are independent random variables with PDF's:*

$$f_y(y) = \alpha y^{\beta} \quad 0 \leq y \leq 1 \tag{37}$$

and

$$f_x(x) = a x^{\beta} (\ln x)^{\rho - 1} \quad 0 \leq x \leq 1 \tag{38}$$

Let $z = xy$.

Then:

$$f_z(z) = \alpha' z^{\beta} (\ln z)^{\rho} \quad 0 \leq z \leq 1 \tag{39}$$

where

$$\alpha' = -\frac{\alpha a}{\rho} \tag{40}$$

Proof of Lemma 1

- Let $w = x$ then:

$$J(x, y) = -w \tag{41}$$

- Therefore the joint probability distribution is:

$$f_{zw}(z, w) = \frac{1}{|J(x, y)|} f_{xy}(w, \frac{z}{w}) \tag{42}$$

- Substituting (41) into (42):

$$f_z(z) = \int \frac{1}{|w|} f_{xy}(w, \frac{z}{w}) dw \tag{43}$$

x and y must be between 0 and 1. Therefore:

$$0 \leq w \leq 1 \tag{44}$$

Proof of Lemma 1

- $$0 \leq \frac{z}{w} \leq 1 \tag{45}$$

- This yields the boundaries of the integral equation as:

$$z \leq w \leq 1 \tag{46}$$

- The independence of the two random variables implies that:

$$f_z(z) = \int_z^1 \frac{1}{w} f_x(w) f_y(\frac{z}{w}) dw \tag{47}$$

- Substituting the PDF's of x and y from equations (37) and (38) into (47):

$$f_z(z) = \int_z^1 \frac{1}{w} a w^{\beta} (\ln w)^{\rho - 1} \alpha \frac{z^{\beta}}{w^{\beta}} dw \tag{48}$$

Proof of Lemma 1

- This can be simplified to:

$$f_z(z) = a\alpha z^{\beta} \int_z^1 (\ln w)^{\rho-1} \frac{1}{w} dw \tag{49}$$

- Finally the solution of the integral equation is:

$$f_z(z) = -\frac{\alpha a}{\rho} z^{\beta} (\ln z)^{\rho} \quad 0 \leq z \leq 1 \tag{50}$$

with:

$$\alpha' = -\frac{\alpha a}{\rho} \qquad \blacksquare \tag{51}$$

PDF of $r_1' r_2' \ldots r_k'$

- r_1' and r_2' are independent.
- The PDF's of r_1' and r_2' have the same form as equations (37) and (38) with $\rho = 1$.
- Using Lemma 1 the PDF of $z_2 = r_1' r_2'$ can be written as:

$$f_{z_2}(z_2) = -\alpha^2 z_2^{\beta} (\ln z_2) \quad 0 \leq z_2 \leq 1 \tag{52}$$

- Equation (52) has the same form as equation (38) with $\rho = 2$.
- The PDF of r_3' has also the same form as equation (38) and r_3' and z_2 are independent.
- Therefore the PDF of $z_3 = z_2 r_3 = r'_1 r'_2 r'_3$ can be found using Lemma 1.

PDF of $r_1' r_2' \ldots r_k'$

- Continuing this process for k times the PDF of $z_k = r_1' r_2' \ldots r_k'$ can be found by:

$$f_{z_k}(z_k) = \frac{(N-1)^k(-1)^{k-1}}{(k-1)!} z_k^{(N-2)} (\ln z_k)^{k-1} \quad 0 \le z_k \le 1 \tag{53}$$

- The CDF of z_k can be obtained by solving:

$$F_{z_k}(z_k) = \int_0^{z_k} f_{z_k}(z_k) dz \tag{54}$$

- The CDF can be computed as:

$$F_{z_k}(z_k) = z_k^{(N-1)} \sum_{i=0}^{k-1} \frac{(-1)^i (N-1)^i}{i!} (\ln z_k)^i \tag{55}$$

Average Rate

- We had:

$$p(\mathbf{k} = k) = p(r_1'r_2'..r_k' \leq \frac{\zeta}{\mathcal{R}^k}) - p(r_1'r_2'...r_{k-1}' \leq \frac{\zeta}{\mathcal{R}^{k-1}}) \tag{56}$$

- Therefore:

$$p(k) = F_{z_k}(\frac{\zeta}{\mathcal{R}^k}) - F_{z_{k-1}}(\frac{\zeta}{\mathcal{R}^{k-1}}) \tag{57}$$

- The average number of stages can be found by:

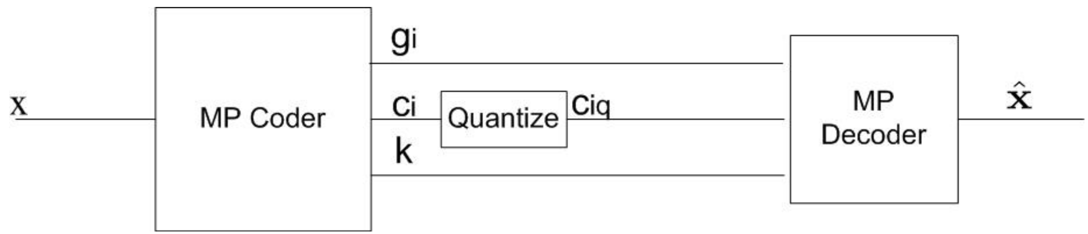
$$E(k) = \sum_{k=1}^{\infty} kp(k) \tag{58}$$

- Therefore the average rate can be calculated by:

$$Rate = E(k)(\log_2 M + \log_2 q)/N \tag{59}$$

Validation

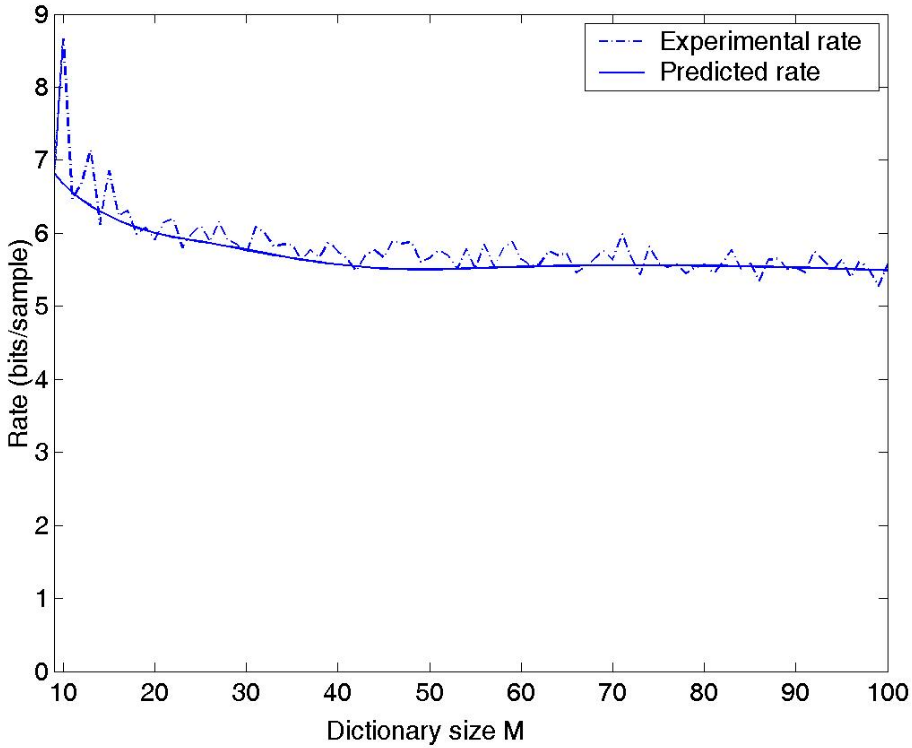
- The Block diagram of the matching pursuit coder used to verify our analysis



- We found the average rate for the optimum dictionary. To verify the results, the dictionary used in simulations must be the optimum dictionary.
- An algorithm similar to LBG algorithm is developed to generate the optimal dictionary.

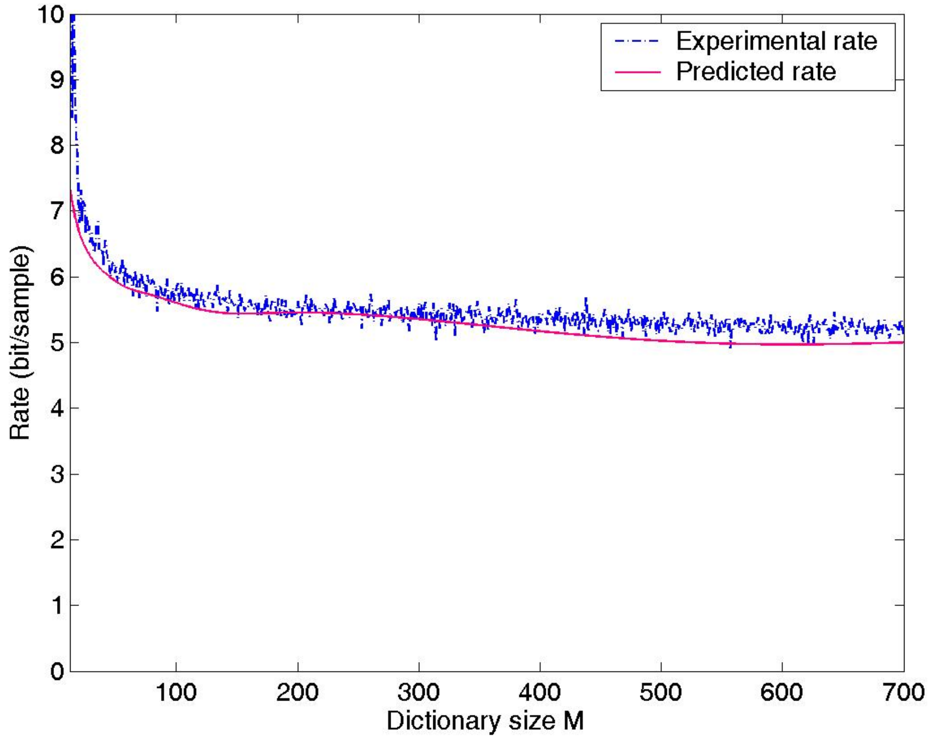
Validation

- Rate for different dictionary sizes for $N = 5$, $q = 16$ and $\zeta = 0.1$:



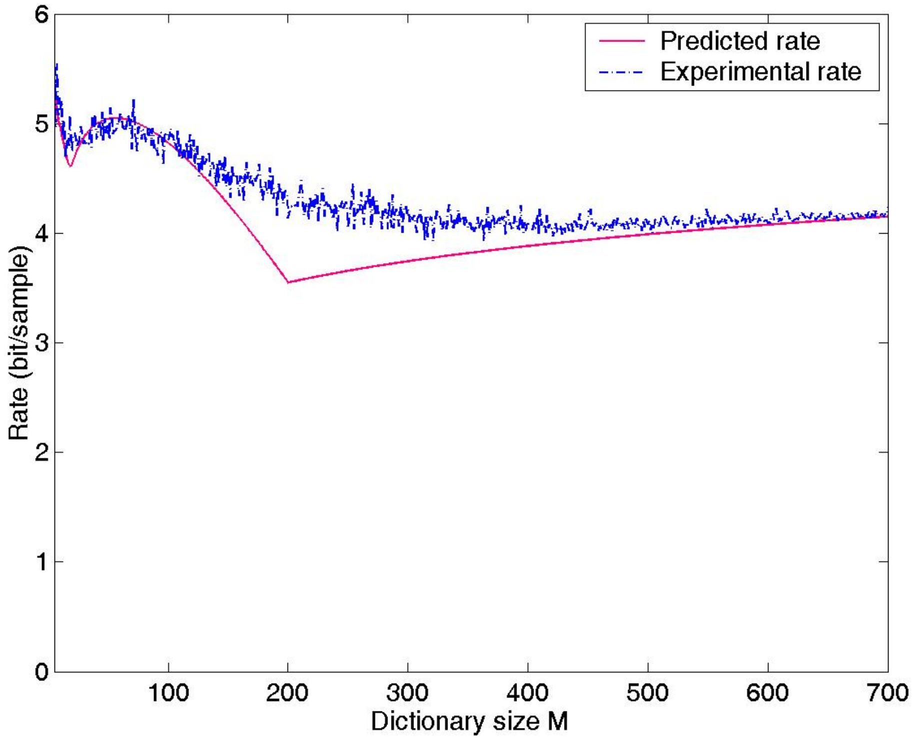
Validation

- Rate for different dictionary sizes for $N = 8, q = 16$ and $\zeta = 0.1$:



Validation

- Rate for different dictionary sizes for $N = 3$, $q = 8$ and $\zeta = 0.1$:



Quantization

- The reconstructed signal is:

$$\hat{x} = \sum_{i=1}^{k} c_i g_{\gamma_i} \tag{60}$$

- Reconstructed signal considering quantization of inner product coefficients:

$$\hat{x}_q = \sum_{i=1}^{k} \hat{c}_i g_{\gamma_i} \tag{61}$$

- Total distortion can be written as:

$$D = D_{MP} + D_Q \tag{62}$$

- The new distortion criterion is:

$$D_{MP} + D_Q \leq E^2 \tag{63}$$

Quantization

- Assume the distribution of the coefficients is uniform in the range of the quantizers and uniform scalar quantizers are used.
- We consider two approaches to quantization of the inner product coefficients.
- In the first approach, the same q level scalar quantizer is used to quantize the inner product coefficients of all stages (fixed range quantization).
- In the second approach the number of quantization levels is kept at q but based on the range of the inner product coefficients of each stage, the decision boundaries of the quantizers are adjusted (variable range quantization).

Fixed Range Quantization

- The quantization distortion can be expressed as:

$$D_Q = \sum_{i=1}^{k} \frac{\|f\|^2}{12q^2} = k\frac{\|f\|^2}{12q^2} \tag{64}$$

and

$$D_{MP} = \|f\|^2 \mathcal{R}^{2k} (r_1' r_2' \ldots r_k')^2 \tag{65}$$

- Substituting (65) and (74) into (63), the distortion criterion can be written as:

$$\|f\|^2 \mathcal{R}^{2k} (r_1' r_2' \ldots r_k')^2 + k\frac{\|f\|^2}{12q^2} \leq E^2 \tag{66}$$

Fixed Range Quantization

- Equation (66) implies that:

$$r_1' r_2' \ldots r_k' \leq \zeta_k' \tag{67}$$

where

$$\zeta_k' = \sqrt{\frac{\zeta^2}{\mathcal{R}^{2k}} - \frac{k}{12q^2\mathcal{R}^{2k}}} \tag{68}$$

and $\zeta = \frac{E}{\|f\|}$.

- The probability of needing k stages to meet the constraint is:

$$p(k) = F_{z_k}(\zeta_k') - F_{z_{k-1}}(\zeta_{k-1}') \tag{69}$$

- Equations (69) and (58) can be used to find the expectation of the number of matching pursuit stages. Once $E(k)$ is found the average rate can be found by equation (59).

Variable Range Quantization

- The inner product coefficients are upper bounded for any dictionary.
- This upper bound exponentially decreases as the stage number increases.
- The quantization range can be decreased according to this upper bound to achieve smaller quantization error.

$$\|R^i f\| = \|f\| \mathcal{R}^i r'_1 r'_2 \ldots r'_i \tag{70}$$

- r'_1 to r'_i are random variables between 0 and 1, therefore:

$$\|R^i f\| \leq \|f\| \mathcal{R}^i \tag{71}$$

- The dictionary vectors are normalized, therefore:

$$|\langle R^i f, g_{\gamma_i} \rangle| \leq \|R^i f\| . \|g_{\gamma_i}\| = \|R^i f\| \tag{72}$$

- Thus:

$$|\langle R^i f, g_{\gamma_i} \rangle| \leq \|f\| \mathcal{R}^i \tag{73}$$

Variable Range Quantization

- Quantization distortion:

$$D_Q = \sum_{i=1}^{k} \frac{\|f\|^2 \mathcal{R}^{2i}}{12q^2} = k\frac{\|f\|^2}{12q^2} \tag{74}$$

$$D_Q = \frac{\|f\|^2(1-\mathcal{R}^{2(k+1)})}{12q^2(1-\mathcal{R}^2)} \tag{75}$$

Therefore the distortion criterion can now be written as:

$$\|f\|^2 \mathcal{R}^{2k}(r_1' r_2' ... r_k')^2 + \frac{\|f\|^2(1-\mathcal{R}^{2(k+1)})}{12q^2(1-\mathcal{R}^2)} \leq E^2 \tag{76}$$

Variable Range Quantization

- This requires:

$$r_1' r_2' \ldots r_k' \le \zeta_k'' \tag{77}$$

where ζ_k'' is defined as:

$$\zeta_k'' \triangleq \sqrt{\frac{\zeta^2}{\mathcal{R}^{2k}} - \frac{1 - \mathcal{R}^{2(k+1)}}{(1 - \mathcal{R}^2) 12 q^2 \mathcal{R}^{2k}}} \tag{78}$$

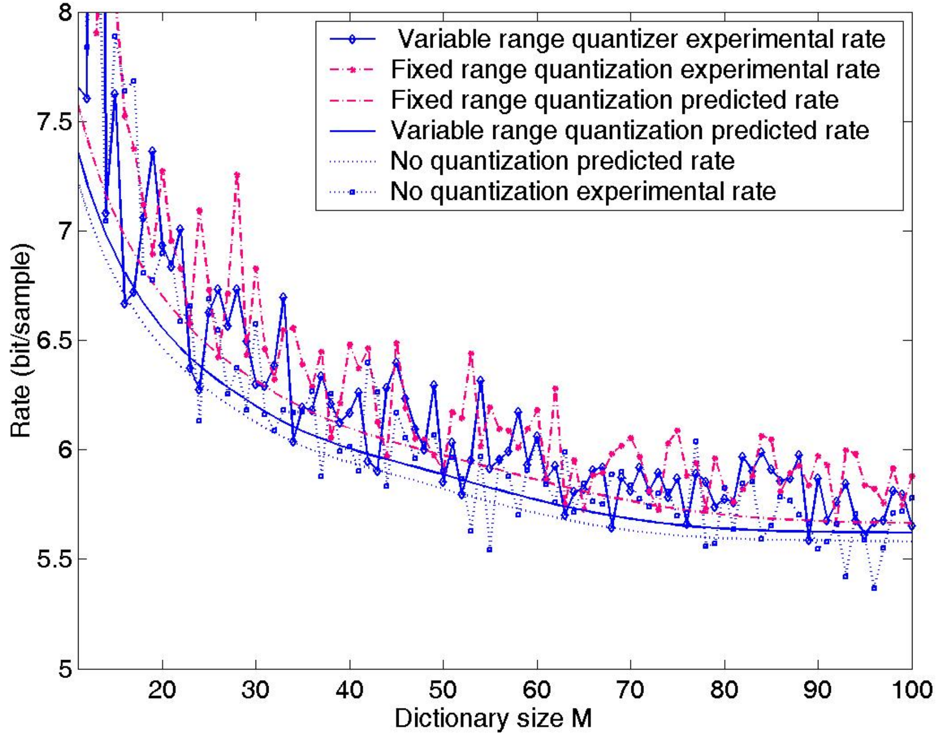
and $\zeta = \frac{E}{\|f\|}$.

- Probability that exactly k stages are required can be found by

$$p(k) = F_{z_k}(\zeta_k'') - F_{z_{k-1}}(\zeta_{k-1}'') \tag{79}$$

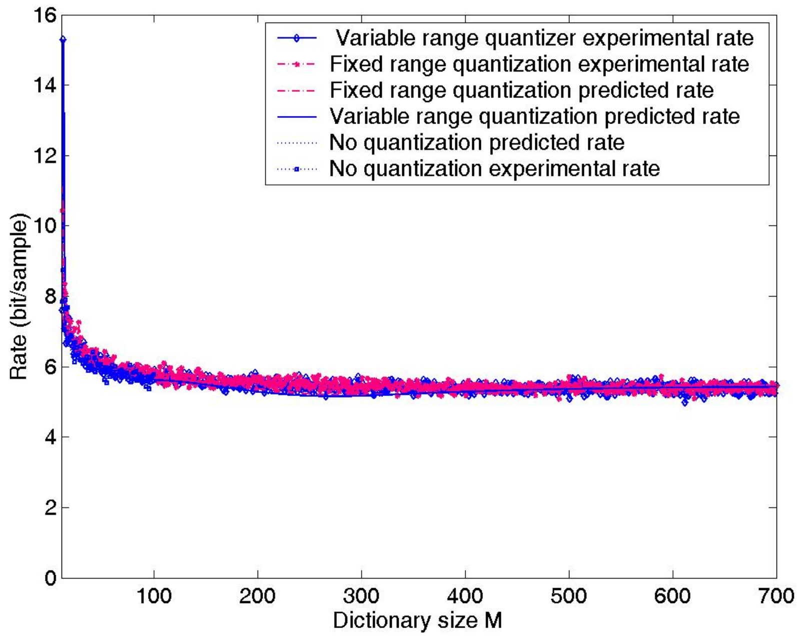
Verification

Rate for $N = 7$, $q = 16$ and $\zeta = 0.1$



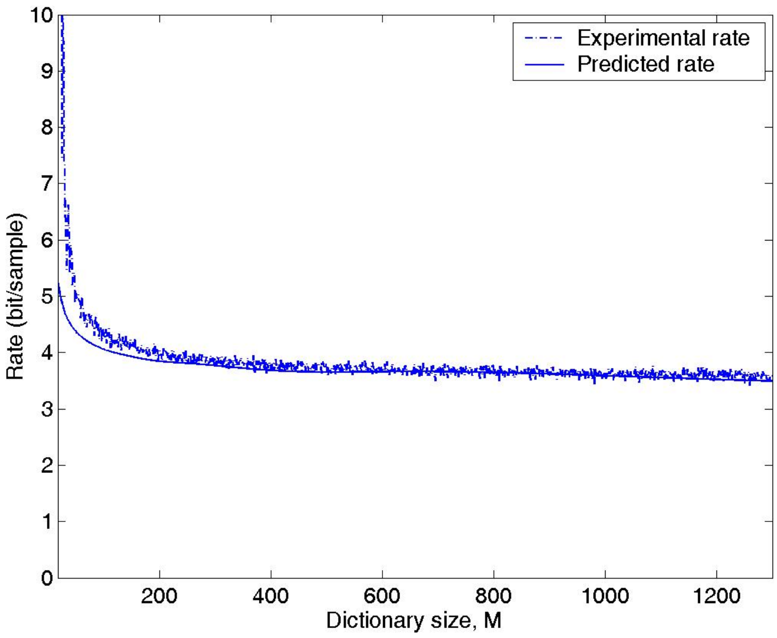
Verification

Rate for $N = 7$, $q = 16$ and $\zeta = 0.1$



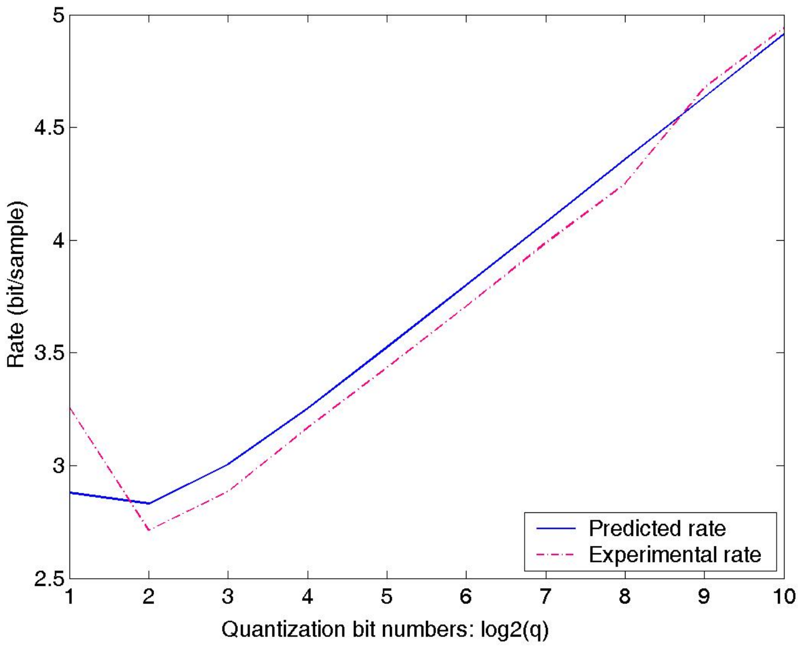
Verification

Rate for $N = 16$, $q = 8$ and $\zeta = 0.2$



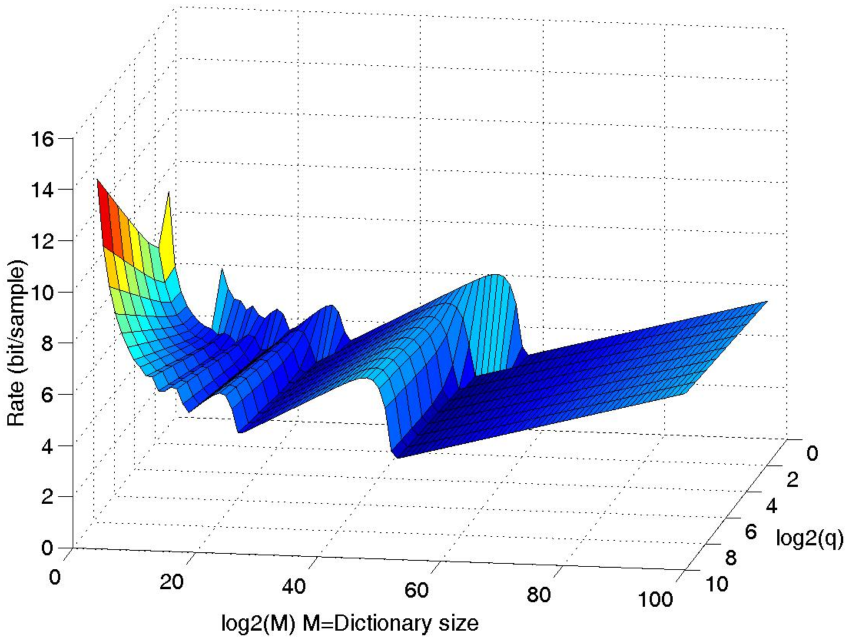
Verification

Rate for $\mathrm{N} = 5$, $M = 200$ and $\zeta = 0.3$



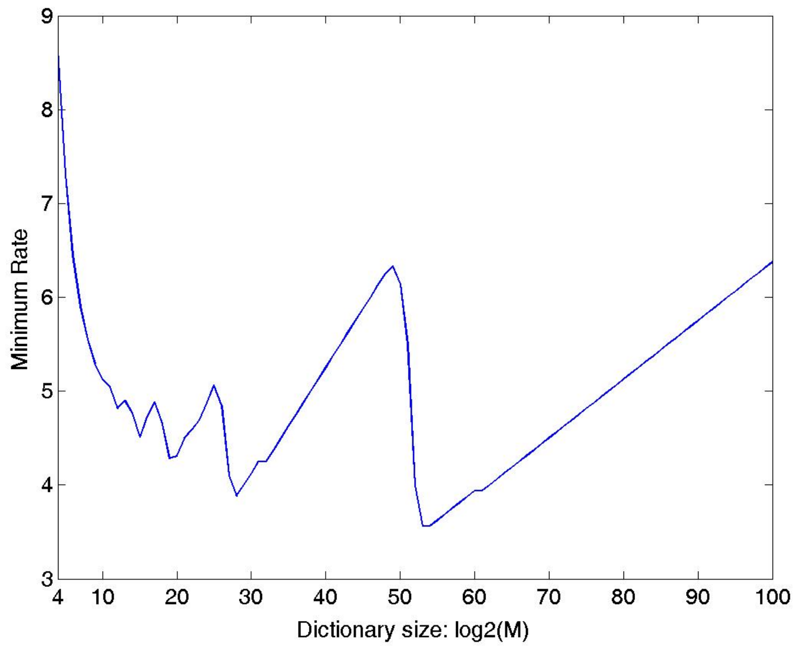
Verification

$N = 16,\ \zeta = 0.1$



Verification

$N = 16,\ \zeta = 0.1$



Future work

- Designing a quantizer with variable step sizes
- Solving the problem for fixed rate and finding the best dictionary size and quantization steps for maximum SNR.
- Finding the Rate Distortion curve of matching pursuit coder

Thank you!