Deterministic and Stochastic, Time and Space Signal Models: An Algebraic Approach

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Electrical & Computer ENGINEERING

Structure and Digital Signal Processing

- **Is DSP algebraic?**
  - By restricting to Linear Algebra are we missing something?
  - Apparently disparate concepts $\Longrightarrow$ instantiations same concept
- **Is DSP geometric?**
  - Constraints may restrict signals to a manifold
  - Algorithms and signal processing should be derived for manifolds
- **Proposed Special Session for ICASSP'06**
  - DSP: Algebra vs. Geometry
- **References for talk:**

Pueschel and Moura, SIAM Journal of Computing, 35:(5), 1280-1316, March 2003
Pueschel and Moura, "Algebraic Theory of Signal Processing, 150 pages, Dec 2004

Algebraic Theory of SP

- **Quick refresh on DSP**
- **DSP: Algebraic view point**
  - Signal Model
- **Algebraic Theory: Time**
  - Time shift
  - Boundary conditions (finite time)
  - Fourier transforms, spectrum
- **Algebraic Theory: Space**
  - Space shift
  - Infinite space: C-transform and DSFT
  - Finite space: DTTs
- **What is it useful for:**
  - Fast algorithms
  - m-D: separable and non-separable, new transforms

DSP

- Scalar, discrete index (time or space) linear signal processing
- 1-D or m-D: indexing set
- Example: infinite discrete time
  - Signals:

$$S(z) = \sum_{n\in\mathbb{Z}} s_n z^{-n} \in \{\sum_{n\in\mathbb{Z}} s_n z^{-n} | (\dots, s_{-1}, s_0, s_1, \dots) \in \ell^2(\mathbb{Z})\} = \mathcal{M}$$

  - Filters:

$$H(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n} \in \{\sum_{n\in\mathbb{Z}} h_n z^{-n} | (\dots, h_{-1}, h_0, h_1, \dots) \in \ell^1(\mathbb{Z})\} = \mathcal{A}$$

  - Convolution (multiplication):

$$Y(z) = H(z) \cdot S(z) = \sum_{n\in\mathbb{Z}} \left(\sum_{i\in\mathbb{Z}} h_i s_{n-i} \right) z^{-n} \in \mathcal{M}$$

  - z-Transform:

$$\begin{aligned} \Phi : \quad & V = \ell^2(\mathbb{Z}) \to \mathcal{M} \\ & \mathrm{s} = (\dots, s_{-1}, s_0, s_1, \dots) \mapsto \textstyle\sum_{n\in\mathbb{Z}} s_n z^{-n} \end{aligned}$$

DSP

- **Fourier Transform: DTFT**

$$S(\omega) \;=\; S(e^{j\omega}) = \sum_{n\in\mathbb{Z}} s_n e^{-j\omega n}; \quad s_n \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega) e^{j\omega n} d\omega$$

- **Spectrum:** $(S(\omega))_{\omega\in W=[-\pi,\pi)}$

- **Impulses:** $E_\omega(z) \;=\; \sum_{n\in\mathbb{Z}} e^{j\omega n} z^{-n}$

- **Eigen property:** $H(z)E_\omega(z) \;=\; H(e^{j\omega n})E_\omega(z)$

- **Linear combination:** $\alpha H(z) + \beta H'(z)$; $\quad \alpha S(z) + \beta S'(z)$

- $\Longrightarrow$ $\mathcal{A}$ and $\mathcal{M}$ are **vector spaces**

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DSP: Algebraic View Point

- **Cascading of filters:** $H(z) \cdot H'(z)$
- $\Longrightarrow$ **makes $\mathcal{A}$ an algebra – the algebra of filters**
- **Convolution (multiplication):** $Y(z) = H(z) \cdot S(z)$
- $\Longrightarrow$ **makes $\mathcal{M}$ an $\mathcal{A}$–module – the module of signals**
- **Signal Model: Triplet** $(\mathcal{A}, \mathcal{M}, \Phi)$
  - where bijective linear mapping $\Phi : V \rightarrow \mathcal{M}$

DSP: Finite Time

- Signals: $S(z)=\sum_{i=0}^{n-1} s_i z^{-i} \in \mathcal{M} \stackrel{?}{=} \{\sum_{i=0}^{n-1} s_i z^{-i}\}$
- Filters: $H(z)=\sum_{i=0}^{n-1} h_i z^{-i} \in \mathcal{A} \stackrel{?}{=} \{\sum_{i=0}^{n-1} h_i z^{-i}\}$
- Convolution (multiplication): $Y(z)=H(z)\cdot S(z) \notin \mathcal{M}$
- Candidates: algebras of filters $\mathcal{A}$ and modules of signals $\mathcal{M}$?

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Algebraic Theory: Shift

- Shift: special type of filter $x \in \mathcal{A}$
  - $h \in \mathcal{A}$ expressed as polynomial or series in x
  - $\Longrightarrow$ x generates $\mathcal{A}$
- Shift invariance: x shift, $\forall s, h:\ h(xs) = x(hs) \Longrightarrow hx = xh$
  - Since ***x*** is shift, $\mathcal{A}$ is commutative, so this is trivially verified
  - Conversely, $\mathcal{A}$ comm., ***x*** generates $\mathcal{A}$, then all filters are shift-inv.
- Which algebras are shift invariant (comm. & generated by single ***x***?)
  - Infinite case: series in ***x*** or polynomials in ***x***
  - Finite dimensional case: *polynomial algebras, p(x)* polyn. deg *n*

$$\mathcal{A} = \mathbb{C}[x]/p(x) = \{\text{polyn. deg } < n, \text{ addition, mult. mod } p(x)\}$$

- Signal Model: finite dimensional case

$$(\mathcal{A} = \mathbb{C}[x]/p(x), \mathcal{M} = \mathbb{C}[x]/p(x), \Phi)$$

$$\Phi : V = \mathbb{C}^n \to \mathcal{M}$$

Algebraic Theory: Infinite Time

- Realization of signal model (infinite time):
  - Time marks and shift operator (Kalman 68): $q \diamond t_n = t_{n+1}$
  - k-fold shift: $q_k \diamond t_n = t_{n+k}, \Longrightarrow q_k = q^k$
  - Linear extension:
    - Extend q from t_n to set $\{\sum s_n t_n\}$
    - Extend from q^k to set of all formal sums $\left\{\sum h_k q^k\right\}$
    - Realization: set $q = x$ and $\diamond = \cdot \Longrightarrow t_{n+1} = x \cdot t_n$
    - Two-term recursion solution: $t_0 = 1, t_n = x^n$
- $\Longrightarrow \mathcal{M} = \{s = \sum s_n x^n\}$ and $\mathcal{A} = \left\{h = \sum h_k x^k\right\}$

$$\Phi : \mathbf{s} \mapsto \sum_{n \in \mathbb{Z}} x^n$$

Remark: we use x rather than z^{-1}

Algebraic Theory: Finite Time

- Realization of signal model (finite time):
  - Problem: $x \cdot x^{n-1} = x^n \quad \notin \mathbb{C}_n[x] = \left\{\sum_{i=0}^{n-1} s_i x^i\right\}$

    $x^{-1} \cdot x^0 = x^{-1} \notin \mathbb{C}_n[x] = \left\{\sum_{i=0}^{n-1} s_i x^i\right\}$
- Boundary condition and signal extension:
  - $x^n = r(x) = \sum_{0 \le k < n} \beta_k x^k$, or $x^n - r(x) = 0$

    Equivalent to right b.c. $s_n = \sum_{0 \le k < n} \beta_k s_k$

    Replaces vector space $\mathbb{C}_n[x]$ by $\mathcal{M} = \mathbb{C}[x]/\left(x^n - r(x)\right)$

$$x^{-1} \in \mathcal{A} \text{ iff } x \not| \; r(x) \Longrightarrow x^{-1} = -\frac{1}{\beta_0}\left(\beta_1 + \beta_2 x + \cdots + \beta_{n-2} - x^{n-1}\right)$$

b.c. ⟹ Right and left signal extension

  - **Signal model**:

$$\left(\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/\left(x^n - r(x)\right), \Phi(\mathbf{s}) = \sum_{0 \le k < n} s_k x^k \in \mathcal{M}\right)$$

Monomial signal extension: $r(x) = a \Longrightarrow x^n = a$

Finite Time and DFT

- Signal model: $\left(\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/\left(x^n - 1\right), \sum_{0 \leq k < n} s_k x^k \in \mathcal{M}\right)$
- Fourier transform: DFT

$$\Delta : \mathcal{M} = \mathbb{C}[x]/\left(x^n - 1\right) \rightarrow \bigoplus_{0 \leq k < n} \mathbb{C}[x]/(x - \omega_n^k)$$

$$\omega_n = e^{-2\pi j/n}$$

  - In matrix format: $\mathrm{DFT}_n = [\omega_n^{k\ell}]_{0 \leq k,\ell < n}$

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Space Signal Model: Space Shift

- Shift: symmetric definition $q \diamond t_n = \left(t_{n+1} + t_{n-1}\right)/2$

  k-fold shift: $q_k \diamond t_n = \left(t_{n+k} + t_{n-k}\right)/2, \Longrightarrow q_k = q^k$

  Differences wrt time model: $q_k \neq q^k, \quad q_k = q_{-k}$

  Lemma: The *k*-fold space shift operator is $q_k = T_k(q)$, the Chebyshev polynomials of the 1st kind

  Linear extension: extend operation of q to

  $$\mathcal{A} = \left\{\sum_{k\geq 0} h_k q_k = \sum h_k T_k(q)\right\}, \quad \mathcal{M} = \{\sum s_n t_n\}$$

  Realization: $q = x$ and $\diamond = \cdot$

  time marks t_n, i.e., satisfy $x \cdot C_n = \left(C_{n+1} + C_{n-1}\right)/2$

  $\Longrightarrow C_n$ sequence of Chebyshev polyn., $\{T_n, U_n, V_n, W_n\}$

Signal Model: Infinite Space

- **Signal Model:**

$$\left(\mathcal{A} = \left\{h = \sum_{k\geq 0} h_k T_k\right\}, \mathcal{M} = \left\{s = \sum_{n\geq 0} s_n C_n\right\}, \Phi\right)$$

- ***C*-transform:** $\Phi : V = \ell^2(\mathbb{N}) \to \mathcal{M} = \left\{\sum_{n\geq 0} s_n C_n\right\}$

$$\Phi(\mathbf{s}) = \sum_{n\geq 0} s_n C_n, \ \ C_n \in \{T_n, U_n, V_n, W_n\}$$

- **Follows from property of Chebyshev polyn.: *k*-fold shift**

$$T_k \cdot C_n = \left(C_{n+k} + C_{n-k}\right)/2$$

- **Fourier transform: DSFT, e.g., choose** $C_n = T_n$

$$\begin{array}{rccl} \Delta : & \mathcal{M} & \to & \left(\bigoplus_{\omega\in[0,\pi]} \mathbb{C}\right) = \mathbb{C}^{[0,\pi]} \\ & s = S(x) & \mapsto & S(\cos\omega)_{\omega\in[0,\pi]} = \omega \mapsto S(\cos\omega). \end{array}$$

Signal Model: Finite Space

- **Left b.c.:** $C_1 = ax + b, a \neq 0, \Longrightarrow C_{-1} = \frac{2-a}{a} C_1 - \frac{2b}{a} C_0$

- **Monomial signal extension:** $C \in \{T, U, V, W\}$

- **Right b.c.: problem with**

$$T_1 \cdot C_{n-1} = x \cdot C_{n-1} = (C_{n-2} + C_n)/2 \notin \left\{ \sum_{0 \geq k < n} s_k C_k \right\}$$

- **Lemma (Monomial right sig. extension): Let** $C \in \{T, U, V, W\}$
Only 4 right bc yield monomial right sig. ext. for $\mathcal{M} = \mathbb{C}[x]/p(x)$

$$C_n = C_{n-2}, C_n = 0, C_n = \pm C_{n-1}$$

$$\Longrightarrow p \in \{C_n - C_{n-2}, C_n, C_n \pm C_{n-1}\}$$

$\Longrightarrow$ 16 possibilities

Finite Sp.Signal Model: Finite C-transf. & DTTs

- **Let seq. Chebyshev poly.:** $C_0, \cdots, C_{n-1}, \ \ C \in \{T, U, V, W\}$
- **Let:** $p \in \{C_n - C_{n-2}, C_n, C_n \pm C_{n-1}\}$

	$C_n - C_{n-2}$	C_n	$C_n - C_{n-1}$	$C_n + C_{n-1}$
T_n	$2(x^2-1)U_{n-2}$	T_n	$(x-1)W_{n-1}$	$(x+1)V_{n-1}$
U_n	$2T_n$	U_n	V_n	W_n
V_n	$2(x-1)W_{n-1}$	V_n	$2(x-1)U_{n-1}$	$2T_n$
W_n	$2(x+1)V_{n-1}$	W_n	$2T_n$	$2(x+1)U_{n-1}$

- **16 finite space signal models:** $(\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/p(x), \Phi)$
- **Finite *C*-transform:** $\Phi : V = \mathbb{C}^n \rightarrow \mathcal{M} = \mathbb{C}[x]/p(x)$

$$\Phi(\mathrm{s}) = \textstyle\sum_{0 \leq k < n} s_k C_k$$

Finite Sp. Sig. Model: Finite C-transf. & DTTs

- Fourier transforms: 16 DTTs (8 DCTs and 8 DSTs)
- Example: Signal model for DCT, type 2
  - Left bc: $s_{-1} = s_0 \Longrightarrow C_1 = C_0 = 1$ afforded by $C = V$
  - Right bc: $s_n = s_{n-1} \Longrightarrow C_n = C_{n-1}$

$$\Longrightarrow p = C_n - C_{n-1} = V_n - V_{n-1} = 2(x-1)U_{n-1}$$

- Sig. model for DCT, type 2:

$$(\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/(x-1)U_{n-1}, \Phi)$$

- DCT, type 2:

$$\Delta : \quad \mathbb{C}[x]/(x-1)U_{n-1} \quad \rightarrow \quad \bigoplus_{0 \leq k < n} \mathbb{C}[x]/(x - \alpha_k)$$

Zeros of $(x-1)U_{n-1},\ \alpha_k = \cos k\pi/n,\ 0 \leq k < n$

$$\text{DCT-2}_n = \text{diag}_{0 \leq k < n}(\cos k\pi/(2n)) \cdot \left[\frac{1}{\cos k\pi/(2n)} \cdot \cos k(\ell + 1/2)\pi/n\right]$$

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Fast Algorithms: DTTs

- **DTTs: DCT, type 2: Direct sum: fast alg. Via poly. factorization**

$$\mathbb{C}[x]/(x-1)U_{n-1} \text{ with basis } b = (V_0, \ldots, V_{n-1})$$

Property of U: $$U_{2m-1} = 2 \cdot U_{m-1} \cdot T_m$$

$$\mathbb{C}[x]/(x-1)U_{2m-1} \cong \mathbb{C}[x]/(x-1)U_{m-1} \oplus \mathbb{C}[x]/T_m$$

$$\text{DCT-2}_{2m} = \mathrm{L}^{2m}_m \cdot (\text{DCT-2}_m \oplus \text{DCT-4}_m) \cdot B.$$

$$B = \begin{bmatrix} 1 & & & & & 1 \\ & \cdot & & & \cdot & \\ & & 1 & 1 & & \\ 1 & & & & & -1 \\ & \cdot & & & \cdot & \\ & & 1 & -1 & & \end{bmatrix} = \begin{bmatrix} \mathrm{I}_m & \mathrm{J}_m \\ \mathrm{I}_m & -\mathrm{J}_m \end{bmatrix} = (\mathrm{DFT}_2 \otimes \mathrm{I}_m)(\mathrm{I}_m \oplus \mathrm{J}_m)$$

Fast Algorithms: DTTs

THEOREM 8.6. *The following recursive algorithms for DTTs are based on the rational factorization* $U_{2m-1} = 2 \cdot U_{m-1} \cdot T_m$. *We also indicate where they first appeared in the literature (to our best knowledge).*

(i) $\text{DCT-1}_{2m+1} = P_{2m+1} \cdot (\text{DCT-1}_{m+1} \oplus \text{DCT-3}_m) \cdot B_{2m+1}$, [27].

(ii) $\text{DST-1}_{2m-1} = P_{2m-1} \cdot (\text{DST-3}_m \oplus \text{DST-1}_{m-1}) \cdot B_{2m-1}$, [55].

(iii) $\text{DCT-2}_{2m} = P_{2m} \cdot (\text{DCT-2}_m \oplus \text{DCT-4}_m) \cdot B_{2m}$, [7].

(iv) $\text{DST-2}_{2m} = P_{2m} \cdot (\text{DST-4}_m \oplus \text{DST-2}_m) \cdot B_{2m}$, [52].

THEOREM 8.7. *The following recursive algorithms for DTTs are based on the rational factorization* $U_{2m} = V_m W_m$.

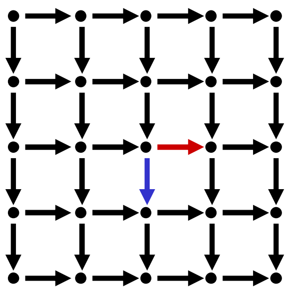
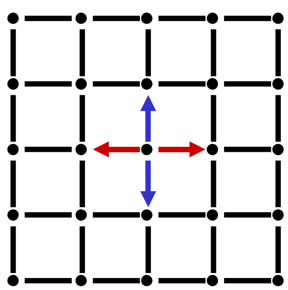
(i) $\text{DCT-1}_{2m} = P_{2m} \cdot (\text{DCT-5}_m \oplus \text{DCT-7}_m) \cdot B_{2m}$.

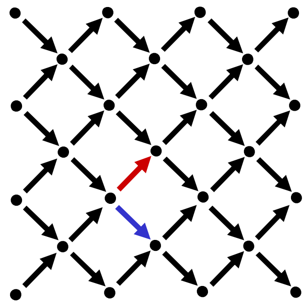
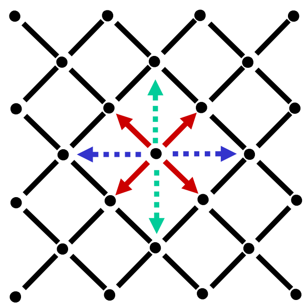
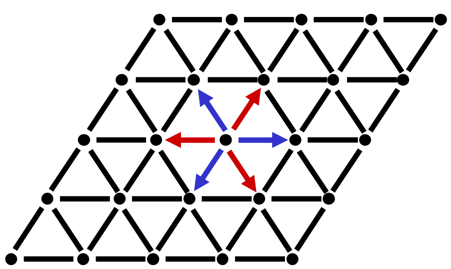
(ii) $\text{DST-1}_{2m} = P_{2m} \cdot (\text{DST-7}_m \oplus \text{DST-5}_m) \cdot B_{2m}$.

(iii) $\text{DCT-2}_{2m+1} = P_{2m+1} \cdot (\text{DCT-6}_{m+1} \oplus \text{DCT-8}_m) \cdot B_{2m+1}$.

(iv) $\text{DST-2}_{2m+1} = P_{2m+1} \cdot (\text{DST-8}_{m+1} \oplus \text{DST-6}_m) \cdot B_{2m+1}$.

Finite Signal Models in Two Dimensions

Visualization (without b.c.)	Signal Model $\mathcal{A} = \mathcal{M}$	Fourier Transform
 time, separable	$\mathbb{C}[x, y]/\langle x^n - 1, y^n - 1\rangle$ time shifts: x, y	$\mathrm{DFT}_n \otimes \mathrm{DFT}_n$
 space, separable	for example $\mathbb{C}[x, y]/\langle T_n(x), T_n(y)\rangle$ space shifts: x, y	$\mathrm{DCT}_n \otimes \mathrm{DCT}_n$ (16 types)

 time, nonseparable	$\mathbb{C}[u,v]/\langle u^n - 1, u^{\frac{n}{2}} - v^{\frac{n}{2}}\rangle$ time shifts: u, v	$\mathrm{DDQT}_{n\times\frac{n}{2}}$ Püschel ICASSP '05 (separable, Mersereau)
 space, nonseparable	$\mathbb{C}[u,v,w]/\langle T_{n/2}(u), T_{n/2}(v), 4w^2-(u+1)(v+1)\rangle$ space shifts: u, v, w	$\mathrm{DQT}_{n\times\frac{n}{2}}$ Püschel ICIP '05
 space, nonseparable	$\mathbb{C}[x,y]/\langle C_n(x,y), \overline{C}_n(x,y)\rangle$ space shifts: u, v	$\mathrm{DTT}_{n\times n}$ Püschel ICASSP '04

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