THE SCHWARZIAN DERIVATIVE AND CONFORMAL MAPPING OF RIEMANNIAN MANIFOLDS

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To Professor Lars V. Ahlfors

1. Introduction. In this paper we identify a tensor which arises in the study of conformal changes of metric on a real Riemannian manifold as a natural generalization of the Schwarzian derivative. The Schwarzian arises in the classical setting of analytic functions and conformal mappings in one complex variable. For a locally injective analytic function \( f \) it is defined by

\[
S(f) = \left( f'' \right)' - \frac{1}{2} \left( f'' \right)^2 = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.
\]

The Schwarzian is important in many areas of complex analysis (see, for example, the recent book of O. Lehto [L]) but it occurs first and foremost through its connection with M"obius transformations. The basic facts are

\[
S(f) = 0 \quad \text{if and only if} \quad f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,
\]

and

\[
S(f \circ h) = S(h) \quad \text{if and only if} \quad f \text{ is M}\ddot{o}bius}.
\]

Equation (1.2) is a special case of a general formula for the Schwarzian of the composite of two analytic functions, which reads

\[
S(f \circ h) = S(h) + (S(f) \circ h)(h')^2.
\]

A generalization of (1.3) will be important for our work.

Let \((M, g)\) be a Riemannian manifold of dimension \( n \geq 2 \) and let \( \nabla \) denote the Riemannian connection for the metric \( g = \langle \ , \ \rangle \). For a smooth function \( \varphi : M \to \mathbb{R} \) we define a tensor

\[
B_{\varphi}(\varphi) = \text{Hess}(\varphi) - d\varphi \otimes d\varphi - \frac{1}{n} \{ \Delta \varphi - \| \text{grad} \ \varphi \|^2 \} g
\]

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where \( \text{Hess}(\varphi) \) is the Hessian of \( \varphi \). Thus, for vector fields \( X \) and \( Y \) on \( M \), we have

\[
B_\varphi(X, Y) = X(Y\varphi) - (\nabla_X Y)(\varphi) - (X\varphi)(Y\varphi) - \frac{1}{n} \{\Delta \varphi - \|\text{grad} \varphi\|^2\} \langle X, Y \rangle.
\]

\( B_\varphi \) is a symmetric \((0, 2)\)-tensor field, and the final term has been chosen to make the trace vanish. We will sometimes omit the subscript \( g \) if the choice of the background metric is clear from the context.

We will always consider \( B_\varphi \) in connection with a conformal metric \( \hat{g} = e^{2\varphi}g \), and we call it the Schwarzian tensor. If \( f: (M, g) \rightarrow (M', g') \) is a conformal local diffeomorphism with \( \hat{g} = e^{2\varphi}g = f^*g' \), \( \varphi = \log \|df\| \), then we define the Schwarzian of \( f \) to be

\[
\mathcal{S}(f) = \mathcal{S}_\varphi(f) = B_\varphi(g).
\]

As a first justification for our terminology, we determine \( \mathcal{S}(f) \) when \( f \) is an analytic function and \( g \) is the Euclidean metric. In this case, \( f^*|dz| = |f'||dz| \), and \( \mathcal{S}(f) = B(\log |f'|) \). Computing in standard coordinates, one finds easily that \( \text{Hess}(\varphi) - d\varphi \otimes d\varphi \) is

\[
\begin{pmatrix}
\text{Re } S(f) - \frac{1}{2} \left| f'' \right|^2 & -\text{Im } S(f) \\
-\text{Im } S(f) & -\text{Re } S(f) - \frac{1}{2} \left| f'' \right|^2
\end{pmatrix}
\]

when we represent the tensor as a \( 2 \times 2 \) matrix. Hence,

\[
\mathcal{S}(f) = \begin{pmatrix}
\text{Re } S(f) & -\text{Im } S(f) \\
-\text{Im } S(f) & -\text{Re } S(f)
\end{pmatrix}.
\]

In Sections 2 and 3 we develop a number of properties of the Schwarzian tensor which follow, fairly directly, from the definition, and which have corresponding formulations in the classical setting. The very first thing we prove is a generalization of the composition formula (1.3) to conformal mappings between Riemannian manifolds. Suppressing the dependence on the metrics, the formula reads

\[
\mathcal{S}(f \circ h) = \mathcal{S}(h) + h^* \mathcal{S}(f).
\]

Without knowledge of (1.3) one would not suspect such a simple relation to hold.

By analogy to the situation for analytic functions, we say that a conformal diffeomorphism \( f: (M, g) \rightarrow (M', g') \) is a Möbius transformation if \( \mathcal{S}_g(f) = 0 \). From (1.6) it follows that the Möbius transformations of a manifold to itself form a group. We denote this group by \( \text{Möb}(M, g) \) or \( \text{Möb}(M) \). It contains the group of homotheties of \( M \) and is contained in the group of all conformal mappings of \( M \).
Along with allowing Möbius transformations between manifolds, the more general point of view here allows us to put another twist on the classical case by defining a conformal metric \( \hat{g} = e^{2\sigma} g \) to be a Möbius metric with respect to \( g \) if \( B_\sigma(\phi) = 0 \). We also say that we have made a Möbius change of metric. This is equivalent to the identity map \( \text{id} : (M, g) \rightarrow (M', e^{2\sigma} g) \) being a Möbius transformation. In Section 2 we find all the Möbius metrics on \( \mathbb{R}^n \) and on \( S^2 \). It is very helpful to have this family of explicit solutions, both for examples and as a guide to more general situations. These include the Poincaré metric on the ball and the spherical metric on \( \mathbb{R}^n \).

The bulk of the paper concerns the two questions of the existence of Möbius metrics on a manifold and the properties of its Möbius group. Our approach in both cases is through a corresponding linear problem. The substitution \( u = e^{-\sigma} \) converts the nonlinear equation \( B_\sigma(\phi) = 0 \) into the linear equation

\[
(1.7) \quad \text{Hess}(u) = \frac{1}{n} (\Delta u)g .
\]

We let \( \mathcal{W}(M) \) denote the space of solutions of (1.7). It is really the entire space of solutions rather than a single solution that is fundamental in our work. Furthermore, we single out a subspace \( \mathcal{W}_K(M) \) of \( \mathcal{W}(M) \) consisting of those \( u \in \mathcal{W}(M) \) for which \( (\Delta u/n) + Ku \) is constant. On \( \mathcal{W}_K(M) \) there is a nondegenerate, indefinite scalar product

\[
\left< u, v \right>_K = u \left( \frac{\Delta u}{n} \right) + v \left( \frac{\Delta u}{n} \right) - \left< \text{grad } u, \text{grad } v \right> + Kuv ,
\]

which seems to occur everywhere in the problems we study. It is not surprising that spaces of constant curvature play a special role. For example, if \( M \) has constant curvature \( K \), then \( \mathcal{W}_K(M) = \mathcal{W}(M) \), and for constant curvature manifolds many of the key features of \( \mathcal{W}(M) \) are already present in the more concrete space \( \mathcal{W}(S^n) \). This is important in our discussion of the Möbius group.

One can use the space \( \mathcal{W}(M, g) \) to realize the metric \( g \) as a warped product. This is the theme of Section 5. In many cases \( \mathcal{W}(M) \) reduces to \( \mathbb{R} \), i.e., all solutions of (1.7) are constant, but not always, and in Theorem 5.4 we give a complete list of the complete manifolds which admit nonconstant solutions to (1.7). This helps to wrap up some problems in Riemannian geometry that have been around for a while, as we shall explain below.

Our approach to the Möbius group in Section 6 is through a natural linear representation of \( \text{Möb}(M) \) into \( \text{Aut}(\mathcal{W}(M)) \). The path of our analysis of the general case follows what we learn by using this representation to study the more familiar group \( \text{Möb}(S^n) \) of Möbius transformations of the sphere. We find, in fact, that in most cases, either \( \text{Möb}(M) \) coincides with the homothety group or that \( M \) is a standard sphere. For the sphere this is a new approach to an old group, we think, and as a benefit we obtain, among other things, formulas generalizing the transformations \( z \leftrightarrow (z + b)/(1 + \bar{b}z) \). We also study infinitesimal Möbius transformations where, once again, the space \( \mathcal{W}(M) \) plays a central role.
Though we look toward two dimensions as a guide, in dimensions three and greater there is an extra feature of the Schwarziian tensor that is absent in the classical case, namely, its relation to Ricci curvature. Under the action of the orthogonal group, the space of curvature tensors decomposes into three summands, the first containing information on the scalar curvature, the second on the Ricci curvature, and the third, consisting of the Weyl conformal curvature tensors. The Schwarziian tensor determines the change in the trace-free part of the Ricci curvature when there is a conformal change in metric. We discuss this in Section 4. This decomposition and the relevant formula bringing in the Schwarziian are well known in the literature, though the fact that it is a generalization of the Schwarziian derivative that is coming in has not been noticed. We give a somewhat different presentation than is usually followed. Our treatment of integrability conditions for the equation $B(\varphi) = p$, where $p$ is a traceless 2-tensor, via a connection on the bundle $R \times TM \times R$, leads to a proof of Weyl's theorem on conformally flat spaces and the vanishing of the Weyl tensor.

The relation between the Schwarziian and the Ricci curvature also allows us to make contact with some problems that go quite a way back. If $(M, g)$ is an Einstein manifold, then a solution of $B_g(\varphi) = 0$ gives a conformal metric $\hat{g} = e^{2\sigma}g$ which is also Einstein. Thus, on an Einstein manifold, the problem of finding solutions to this equation is essentially the problem of finding conformal Einstein metrics. This problem was studied by H. Brinkman [Br] in 1925. He also discovered that the local geometry determined by a single solution is that of a warped product, though this terminology was not used then. This local structure has been rediscovered many times, and only A. Fialkow [F] in 1939 mentions briefly the possibility of considering multiple solutions. In Section 5 we have tried to give an accurate and a fair accounting of what is a somewhat tangled and, unfortunately, sometimes an unreliable succession of papers. And while we do not wish to emphasize any one particular problem, we do give in Theorem 5.5 a list of the complete Einstein manifolds admitting a conformal, nonhomothetic Einstein metric, finishing the story.

Closer in spirit to the present paper are papers of Yano [Yn] and Tashiro [T1], [T2]. In a series of short papers, Yano considered conformal transformations mapping circles to circles, which he called concircular. His main results are a consequence of the work in Section 3 where we study the relations between the Schwarziian and the geometry of submanifolds. Tashiro considers an infinitesimal approach to Yano's concircular transformations, but his work is marred by errors. In this paper we sincerely hope to catch the interest of readers who know all the words in the title. With this in mind we have not hesitated to include commentary and background material when it might prove helpful, and we have tried to strike a balance between a coordinate free approach and explicit calculations in local coordinates. We do feel that the point of view we have tried to develop here has led to a greater degree of clarity and unity in the subject. Since we began our work, there have been a number of developments. A (later) paper of ours [OS1] has appeared in which we generalize a theorem of Nehari [N] relating the growth of the Schwarziian derivative of a conformal mapping to its injectivity. These ideas
are extended in the thesis of M. Chuaqui [C]. K. Carne, who had been interested in these problems and who read an earlier version of this paper, has developed his ideas on the subject in [Ca]. A paper of L. Ahlfors [A], written some time ago, has now appeared. Some connections between Ahlfors’s work and ours are studied in Chuaqui’s thesis. We also thank L. Ahlfors, C. Epstein, R. Schoen, and M. Wolf, among others, for their interest and encouragement.

For our part we now feel that there is another and perhaps even a more geometric setting for the Schwarzian and its generalization. In [OS2] we define the Möbius connection on the bundle of conformal 2-jets from the manifold into the sphere, and we show that the Schwarzian can be realized as the difference of two such connections when there is a conformal change of metric.

2. Elementary properties. In this section we develop some elementary properties of the Schwarzian tensor. We introduce Möbius transformations between manifolds and the Möbius group of a manifold, and make a preliminary study of the integrability of the equation $B(\varphi) = 0$ in relation to the local geometry of the manifold.

Basic to this program is the fact that the tensor $B(\varphi)$, though nonlinear in $\varphi$, does have an additivity property when there is a corresponding conformal change in the background metric.

**Lemma 2.1.** Let $\varphi, \sigma: M \to \mathbb{R}$ be smooth functions on $(M, g)$. Then

$$B_\varphi(\varphi + \sigma) = B_\varphi(\varphi) + B_\varphi(\sigma)$$

where $\hat{g} = e^{2\sigma}g$.

**Proof.** A standard formula gives the relation between the Riemannian connections for $g$ and $\hat{g}$ as

$$(2.0) \quad \hat{\nabla}_X Y = \nabla_X Y + (X \varphi)Y + (Y \varphi)X - \langle X, Y \rangle \text{grad } \varphi$$

where $g = \langle \cdot, \cdot \rangle$ and grad is the gradient with respect to $g$. Therefore,

$$\text{Hess}_g(\sigma)(X, Y) = XY\sigma - (\hat{\nabla}_X Y)\sigma$$

$$= \text{Hess}_g(\sigma)(X, Y) - (X \varphi)Y\sigma - (Y \varphi)X\sigma + \langle X, Y \rangle \langle \text{grad } \varphi, \text{grad } \sigma \rangle.$$ 

By definition, $B_\sigma(\sigma)$ is obtained from $\text{Hess}_g(\sigma) - d\sigma \otimes d\sigma$ by subtracting that multiple of $\hat{g}$ which makes the contraction of $B_\varphi(\sigma)$ vanish with respect to $\hat{g}$. Since $\hat{g} = e^{2\sigma}g$, it is easy to see that this is equivalent to subtracting from $\text{Hess}_g(\sigma) - d\sigma \otimes d\sigma$ that multiple of $g$ which will make the contraction of $B_\sigma(\sigma)$ vanish with respect to $g$.

Having observed this, we find from the above that

$$B_\sigma(\sigma) = B_\sigma(\sigma) - d\varphi \otimes d\sigma - d\sigma \otimes d\varphi + \frac{2}{n} \langle \text{grad } \varphi, \text{grad } \sigma \rangle g.$$
In particular, since
\[
\text{Hess}_\varphi(\varphi + \sigma) - d(\varphi + \sigma) \otimes d(\varphi + \sigma)
\]
\[
= \text{Hess}_\varphi(\varphi) - d\varphi \otimes d\varphi + \text{Hess}_\varphi(\sigma) - d\sigma \otimes d\sigma - d\varphi \otimes d\sigma - d\sigma \otimes d\varphi,
\]
we obtain
\[
B_\varphi(\varphi + \sigma) = B_\varphi(\varphi) + B_\varphi(\sigma)
\]
as desired.

For analytic functions \(f\) and \(h\), one has the important identity
\[
(2.1) \quad S(f \circ h) = S(h) + (S(f) \circ h)(h')^2,
\]
often attributed to Cayley. Now let \(h: (M, g) \to (M', g')\), \(f: (M', g') \to (M'', g'')\) be conformal diffeomorphisms (or immersions) with \(h^*g' = e^{2\varphi}g\) and \(f^*g'' = e^{2\varphi}g'\). Then \((f \circ h)^*g'' = e^{2(f \circ h)}\varphi\), and the lemma yields
\[
(2.2) \quad S_g(f \circ h) = S_g(h) + h^*S_g(f);
\]
in fact, (2.2) is equivalent to the lemma. One may easily derive (2.1) from (2.2) through equality of the real and imaginary parts using (1.5). Thus, the fact that \(h'\) enters in (2.1) to the second power is because in computing the effect of the pullback on \(S_g(f)\) one applies \(h'_{\varphi}\) to a pair of tangent vectors.

Recall that Möbius transformations are conformal diffeomorphisms with vanishing Schwarzian. From (2.2), \(S(f \circ h) = S(h)\) for all conformal mappings \(h\) if and only if \(f\) is Möbius. Clearly, \(\text{id}: (M, g) \to (M, g)\) is Möbius as is any homothety (a conformal diffeomorphism whose derivative has constant norm). By (2.2)
\[
(2.3) \quad S(f^{-1}) = -(f^{-1})^*S(f)
\]
for any conformal diffeomorphism \(f\). Finally, let \(f: (M, g) \to (M, g)\) be a conformal diffeomorphism, let \(\hat{\varphi} = e^{2\varphi}g\) (here \(\hat{\varphi}\) is not necessarily \(f^*\varphi\)), and consider the composition of conformal maps
\[
(M, g) \xrightarrow{h} (M, \hat{\varphi}) \xrightarrow{f} (M, \hat{\varphi}) \xrightarrow{h^{-1}} (M, g)
\]
where \(h\) is the identity. Then
\[
S_g(f) = S_g(h^{-1}fh) = S_g(h) + h^*(S_g(f) + f^*S_g(h^{-1}))
\]
\[
= S_g(h) + S_g(f) - f^*S_g(h),
\]
from (2.2) and (2.3). If \( B_\phi(\sigma) = 0 \), which is to say that \( S_\phi(h) = 0 \) or that we have made a Möbius change of metric, then \( \mathcal{J}_\phi(f) = \mathcal{J}_\phi(f) \).

We summarize these remarks as the following theorem.

**Theorem 2.2.** Composites and inverses of Möbius transformations are Möbius. The Möbius transformations of Riemannian manifold \((M, g)\) into itself form a group \( \text{Möb}(M) = \text{Möb}(M, g) \). \( \text{Möb}(M, g) \) contains the group of homotheties \( \text{Hty}(M, g) \), is contained in the group of conformal transformations \( \text{Conf}(M, g) \), and is unchanged by a Möbius change of metric.

Note that \( \text{Möb}(M) \) is a closed subgroup of the Lie group \( \text{Conf}(M) \) and so is itself a Lie group. Its precise relation to \( \text{Hty}(M) \) and \( \text{Conf}(M) \) will be described in Section 6.

Next, we consider the equation

\[(2.4) \quad B(\phi) = p\]

where \( p \) is a field of symmetric \((0, 2)\) tensors of trace 0. This is a nonlinear equation, but the additivity formula of Lemma 2.1 implies that, if \( \phi_0 \) is a particular solution to (2.4) and if \( \tilde{g} = e^{2\phi_0}g \), then the problem of determining all solutions reduces to solving the homogeneous equation \( B_\phi(\sigma) = 0 \) on the conformally equivalent manifold \((M, \tilde{g})\).

We say that (2.4) is **fully integrable** at \( x \in M \) if for every \( X \in T_xM \) there exists a locally defined solution with \( \text{grad} \phi(x) = X \). For the present we shall be interested primarily in the homogeneous equation

\[(2.5) \quad B(\phi) = 0.
\]

In Section 5 we give a detailed analysis of the local structure of multiple solutions to (2.5), bringing in the geometry of warped products. Here, we prove the following theorem.

**Theorem 2.3.** Let \((M, g)\) have dimension \( \geq 2 \). The equation \( B(\phi) = 0 \) is fully integrable at \( x \in M \) if and only if \( M \) has constant curvature near \( x \).

Before beginning the proof, we make a few general remarks on the equation. It is useful to make the substitution \( u = e^{-\phi} \) in (2.5), for then

\[
\text{Hess}(u)(X, Y) = \langle \nabla_X \text{grad} e^{-\phi}, Y \rangle
= -e^{-\phi}\{\text{Hess}(\phi)(X, Y) - (X \phi) Y\phi\},
\]

and (2.5) becomes the linear equation

\[(2.6) \quad \text{Hess}(u) = \dot{\lambda} g, \quad \dot{\lambda} = \Delta u/n.\]
The nonhomogeneous equation $B(\varphi) = p$ becomes

\[ (2.7) \quad \text{Hess}(u) + up = \lambda g. \]

Observe also that (2.6) may be written

\[ (2.6)' \quad \nabla X \text{ grad } u = \lambda X \]

for any vector field $X$ and, likewise, that (2.5) may be written

\[ (2.5)' \quad \nabla X \text{ grad } \varphi - (X \varphi) \text{ grad } \varphi = vX, \quad v = (\Delta \varphi - \|\text{grad } \varphi\|^2)/n. \]

The substitution above is suggested by regarding (2.4) as a Riccati equation, precisely as one does in solving $S(f) = p$ in the classical setting. One final word about this: the local analyses of (2.5) and (2.6) are equivalent, as least for compact families of solutions, for we can always add a large constant to a solution to $\text{Hess}(u) = \lambda g$ to guarantee that we can form $\varphi = -\log u$ in a given neighborhood. There may, however, be nonconstant global solutions of (2.6) but none of (2.5); see Section 5. That is, globally, the substitution goes only one way. In any event, we let $U(M)$ denote the set of all solutions to (2.6).

We prove the necessity in Theorem 2.3 first. Suppose that the equation $B(\varphi) = 0$ is fully integrable at $x \in M$ and let $\varphi$ be any solution such that $\text{grad } \varphi(x) \neq 0$. The level surfaces of $\varphi$ then foliate a neighborhood of $x$. The special features of this foliation, deriving from our particular equation, are described in the following lemma.

**Lemma 2.4.** Let $N$ be the unit vector field in the direction of $\text{grad } \varphi$ in a neighborhood of $x$. For $y$ near $x$, the sectional curvature of every plane containing $N$ has the same value $K(y)$, and $K$ is constant on each leaf of the foliation.

**Proof of lemma.** As in (2.5)', let $X$ be a vector field and write $B(\varphi) = 0$ in the form

\[ (2.8) \quad \nabla X \text{ grad } \varphi - (X \varphi) \text{ grad } \varphi = vX. \]

Then

\[
X \|\text{grad } \varphi\| = \langle \nabla X \text{ grad } \varphi, N \rangle
= \langle X, N \rangle (v + \|\text{grad } \varphi\|^2).
\]

If we take $X \perp N$, we see that $\|\text{grad } \varphi\|$ is constant on each leaf, and thus from (2.8)

\[
\nabla_X \langle \text{grad } \varphi \rangle = \frac{v}{\|\text{grad } \varphi\|} X, \quad X \perp N.
\]
On the other hand, with $X = N$ we obtain
\[ v = N \| \text{grad } \varphi \| - \| \text{grad } \varphi \|^2. \]

Note also that $\nabla_v N = 0$ since $\nabla_{\text{grad } \varphi} \text{grad } \varphi$ is a multiple of $\text{grad } \varphi$. Thus for $X \perp N$ again, we have
\[ \langle [X, N], N \rangle = \langle \nabla_X N - \nabla_N X, N \rangle = 0 - N \langle X, N \rangle = 0, \]
so that
\[ Xv = X(N \| \text{grad } \varphi \| - \| \text{grad } \varphi \|^2) \]
\[ = [X, N] \| \text{grad } \varphi \| + N(X \| \text{grad } \varphi \|) = 0. \]

We conclude that $v$ is also constant on each leaf.

Now let $X \in T_y M$ be a unit vector orthogonal to $N_y$ and extend $X$ to be orthogonal to $N$ everywhere. The plane spanned by $X_y$ and $N_y$ then has sectional curvature
\[ K = \langle \nabla_X \nabla_N N - \nabla_N \nabla_X N - \nabla_{[X, N]} N, X \rangle \]
\[ = -\langle \nabla_N \left( \frac{v}{\| \text{grad } \varphi \|} X \right) + \frac{v}{\| \text{grad } \varphi \|} [X, N], X \rangle \]
\[ = -N \left( \frac{v}{\| \text{grad } \varphi \|} \right) \left( \frac{v}{\| \text{grad } \varphi \|} \right) - \frac{v}{\| \text{grad } \varphi \|} \langle \nabla_X N, X \rangle \]
\[ = -N \left( \frac{v}{\| \text{grad } \varphi \|} \right) - \left( \frac{v}{\| \text{grad } \varphi \|} \right)^2. \]

Therefore, $K$ depends upon $y$ alone and not on the vector $X_y$. Moreover, if $Y \perp N$, another application of (2.9) shows that $Y K = 0$. Thus, $K$ is constant on each leaf of the foliation, completing the proof of the lemma.

The necessity in Theorem 2.3 now follows quickly. When $\dim M = 2$, consider two solutions $\varphi_1, \varphi_2$ of $B(\varphi) = 0$ such that $\text{grad } \varphi_1(x)$ and $\text{grad } \varphi_2(x)$ are linearly independent. Since the foliations corresponding to $\varphi_1$ and $\varphi_2$ are then transverse and since the Gaussian curvature is constant on leaves of either, it is constant near $x$. When $\dim M \geq 3$, it suffices to show that when $y$ is near $x$ the sectional curvature of all planes through $y$ coincide. This holds when $y = x$ since the lemma implies that any planes in $T_y M$ with nontrivial intersection will have the same sectional curvature. Thus, the problem now reduces to showing that $B(\varphi) = 0$ is fully integrable at all points $y$ near $x$. But it is easy to see that the equation $B(\varphi) = 0$ is fully
integrable at \( y \) if and only if the corresponding linear equation \( \text{Hess}(u) = \lambda g \) is fully integrable at \( y \), and for a linear equation full integrability is clearly an open condition.

The proof of sufficiency in Theorem 2.3 requires two facts, which will also be of use in Section 5.

**Lemma 2.5.** \( B(\varphi) = 0 \) is fully integrable on \( \mathbb{R}^n \) with respect to the euclidean metric. The set \( \mathcal{U}(\mathbb{R}^n) \) consists of all functions of the form

\[
u(x) = a|x|^2 + b \cdot x + c, \quad a, c \in \mathbb{R}, \quad b \in \mathbb{R}^n.
\]

Here, we write \( b \cdot x \) for the usual euclidean inner product. As above, a positive function \( u \in \mathcal{U}(M) \) gives a solution to \( B(\varphi) = 0 \) with \( \varphi = -\log u \).

**Proof of Lemma 2.5.** Equation (2.6) becomes

\[\partial_i \partial_j u = \partial_j \partial_i u, \quad i \neq j.\]

This implies \( \partial_i \partial_j \partial_i u = 0 \); so \( u = f(x_2, \ldots, x_n)x_1^2 + h(x_2, \ldots, x_n)x_1 + k(x_2, \ldots, x_n) \). Since \( \partial_i \partial_j u = 0, j \neq 1, f \) and \( h \) are constants. Applying this argument to all indices, we obtain

\[u = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i=1}^{n} b_i x_i + c.\]

Since \( \partial_i \partial_j u = \partial_j \partial_i u \), all the coefficients \( a_i \) coincide proving that any function in \( \mathcal{U}(\mathbb{R}^n) \) has the form in (2.10). The function \( \varphi = -\log u \) then satisfies \( B(\varphi) = 0 \). The form of the solution also makes it clear that we can produce a \( \varphi \) with a specified gradient. Lemma 2.5 is proved.

**Lemma 2.6.** If \( B(\varphi) = 0 \) in \( \mathbb{R}^n \), then the metric \( e^{2\varphi}(\text{euc}) \) has constant curvature \( K_\varphi \), and \( \varphi \) can be chosen so that \( K_\varphi \) is any given value.

**Proof.** The sectional curvature of a metric \( e^{2\varphi} \) (euc) of the plane spanned by \( \partial/\partial x_i, \partial/\partial x_j \) is given by

\[K_{ij} = e^{-2\varphi} \{ -\partial_i \partial_j \varphi + (\partial_i \varphi)^2 - \partial_j \partial_i \varphi + (\partial_j \varphi)^2 - |\text{grad } \varphi|^2 \}.\]

With \( u = e^{-\varphi} \) this becomes

\[K_{ij} = u(\partial_i \partial_j u + \partial_j \partial_i u) - |\text{grad } u|^2;\]

see also Section 5. If \( u \) is as in (2.10), then

\[K_{ij} = 4ac - |b|^2 = \frac{2u(0)A_0}{n} - |\text{grad } u(0)|^2.\]
We see from this that we get the same value for any plane, at any point, which after a rotation of $\mathbb{R}^n$, becomes parallel to one of the coordinate planes. As we can do this for any plane, it follows that $e^{2\varphi}(\text{euc}) = u^{-2}(\text{euc})$ has constant curvature

$$K = 4ac - |b|^2.$$  

From Lemma 2.5 and 2.6 we can deduce the sufficiency in Theorem 2.3 as follows. Suppose that $(M, g)$ has constant curvature $K$ near a point $x \in M$. Choose $\varphi = -\log(a|x|^2 + b \cdot x + c)$, solving $B(\varphi) = 0$ on $\mathbb{R}^n$ such that $g_1 = e^{2\varphi}(\text{euc})$ has curvature $K$, and let $f$ be a local isometry of a neighborhood of $x$ into $(\mathbb{R}^n, g_1)$. If we establish full integrability for $B_{g_1}(\sigma) = 0$ on $(\mathbb{R}^n, g_1)$, the same will then be true for $(M, g)$ at $x$. By the transformation formula, Lemma 2.1, $B(\varphi + \sigma) = B(\varphi) + B_{g_1}(\sigma)$, where the background metric is the euclidean metric. Hence to solve $B_{g_1}(\sigma) = 0$, and indeed to do so with prescribed initial gradient, we need only take $\sigma$ to be of the form

$$(2.11) \quad \sigma(x) = -\log \left( \frac{A|x|^2 + B \cdot x + C}{a|x|^2 + b \cdot x + c} \right), \quad A, C \in \mathbb{R}, \quad B \in \mathbb{R}^n.$$  

This completes the proof of Theorem 2.3.

Observe from (2.10) that the solutions to $B(\varphi) = 0$ on $\mathbb{R}^n$ include metrics homothetic to the standard euclidean metric, the spherical metric on $\mathbb{R}^n$,

$$(2.12) \quad \frac{2}{1 + |x|^2}(\text{euc}),$$

and the Poincaré metric for a ball or a half-space, say

$$(2.13) \quad \left( \frac{2}{1 - |x|^2} \right)^2(\text{euc}) \quad \text{or} \quad \frac{1}{x_2^n}(\text{euc}).$$

That is, these are all examples of Möbius metrics on $\mathbb{R}^n$ or on domains in $\mathbb{R}^n$. In fact, one can say more. Suppose we insist that $\varphi$, as given in (2.10), is to be defined on all of $\mathbb{R}^n$. Then there are essentially two possibilities for the metric $g_1 = e^{2\varphi}(\text{euc})$. If $a = 0$, then $b = 0$ and $c > 0$ so that $g_1$ is homothetic to the euclidean metric. If $a \neq 0$, then $a, c > 0$, and after a translation of $\mathbb{R}^n$ by $b/2a$, $\varphi$ has the form

$$\varphi(x) = -\log \left( a|x|^2 + \frac{K}{4a} \right)$$

where $K = 4ac - |b|^2$ is the curvature of $g_1$, as above. So $K > 0$, and further homotheties (which preserve the vanishing of $B(\varphi)$) will bring $g_1$ into the form (2.12). Similar reasoning leads to the metrics (2.13) if we insist on finding complete Möbius metrics on a ball or on a half-space.
More generally, we note that in the proof of the sufficiency in Theorem 2.3 we have also determined \( \mathcal{U}(\mathbb{R}^n, g_1) \) where \( g_1 = (a|x|^2 + b \cdot x + c)^{-2} \) (euc). It consists of the functions \( u = e^{-\sigma} \) where \( \sigma \) is as in (2.11). In local calculations it is quite useful to have these explicit solutions.

3. The Schwarzian and the second fundamental form. In this brief section we want to illustrate the usefulness of isolating the Schwarzian tensor when computing the change in geometric quantities under a conformal change of metric. The motivation here comes very much from the classical case. Möbius transformations in the plane, after all, are characterized not only by the vanishing of the Schwarzian derivative but also by the fact that they map circles to circles. This turns out to be a special instance of the role of \( B(\phi) \) in relating the second fundamental forms of a submanifold with respect to conformal metrics \( g \) and \( \hat{g} = e^{2\phi} g \). Generalizing the classical case in this way gives, among other things, a simple and invariant formulation of some of the results in four papers of Yano [Yn] on concircular transformations.

To fix notation, let \( g = \langle \ , \ \rangle \) be a metric on \( M \), let \( P \) be a submanifold of \( M \), and let \( NP \subset T_p M \) be its normal bundle. Tangent vector fields along \( P \) will be denoted \( X, Y, \) etc.; normal vector fields will be denoted \( A, B, \) etc. The Riemannian connection for \( P \) will be denoted by \( \nabla' \) and the normal connection by \( \nabla'' \); \( \nabla \) is the Riemannian connection for \( M \). Then

\[
\begin{align*}
(a) \quad \nabla_X Y &= \nabla'_X Y + s(X, Y), \\
(b) \quad \nabla_X A &= t(X, A) + \nabla''_X A,
\end{align*}
\]

where \( s \) is the (vector-valued) second fundamental form of \( P \) in \( M \) and the tensor \( t \) satisfies

\[
\begin{align*}
(c) \quad \langle t(X, A), Y \rangle &= -\langle A, s(X, Y) \rangle.
\end{align*}
\]

Now consider a conformal metric \( \hat{g} = e^{2\phi} g \). A straightforward calculation using (2.0) and (3.1) leads to the relations (in obvious notation)

\[
\begin{align*}
(a) \quad \nabla'_X Y &= \nabla'_X Y + (X \phi) Y + (Y \phi) X - \langle X, Y \rangle (\text{grad } \phi)^T, \\
(b) \quad s(X, Y) &= s(X, Y) - \langle X, Y \rangle (\text{grad } \phi)^N, \\
(c) \quad \hat{t}(X, A) &= t(X, A) + (A \phi) X, \\
(d) \quad \nabla''_X A &= \nabla''_X A + (X \phi) A, 
\end{align*}
\]

where \( (\text{grad } \phi)^T \) and \( (\text{grad } \phi)^N \) are the tangential and normal parts of \( \text{grad } \phi \).
The connections $V'$ and $V''$ determine a covariant derivative of $s$ as

$$(3.3) \quad (V_Z s)(X, Y) = V'_Z s(X, Y) - s(V'_Z X, Y) - s(X, V'_Z Y).$$

We want to compare this with $(\tilde{V}_Z s)(X, Y)$. Again, this is a direct, if somewhat lengthy, calculation using (3.2), and we will not reproduce the intermediate steps. The final formula becomes nicely symmetric if in one term in the calculation one uses

$$\nabla_Z (\text{grad} \, \phi)^N = \{\nabla_Z (\text{grad} \, \phi - (\text{grad} \, \phi)^T)\}_N$$

$$= (\nabla_Z \text{grad} \, \phi)^N - s(Z, (\text{grad} \, \phi)^T).$$

We then find that

$$(3.4) \quad (\tilde{V}_Z s)(X, Y) = (V_Z s)(X, Y) + \alpha(X, Y, Z) - \langle X, Y \rangle \{M(\phi)Z\}_N,$$

where

$\alpha(X, Y, Z) = s((\text{grad} \, \phi)^T, \langle X, Y \rangle Z + \langle Y, Z \rangle X + \langle Z, X \rangle Y)$$

$- ((Z\phi)s(X, Y) + (X\phi)s(Y, Z) + (Y\phi)s(Z, X))$

and

$$(3.5) \quad M(\phi)Z = \nabla_Z \text{grad} \, \phi - (Z\phi) \text{grad} \, \phi - \frac{1}{n} \{\Delta \phi - \|\text{grad} \, \phi\|^2\} Z.$$

Observe that $\alpha$ is symmetric in its three arguments and that, at any point $x$, $M(\phi)$ is the linear transformation of $T_x M$ which represents $B(\phi)$, i.e.,

$$(3.6) \quad \langle M(\phi)Z, W \rangle = B(\phi)(Z, W), \quad Z, W \in T_x M.$$

Equation (3.4) is the main formula in this section. We now wish to draw some conclusions in special cases. First, suppose that $M$ is a surface and let $\Gamma$ be a curve on $M$. Choose unit tangent and unit normal vector fields (with respect to $q$) $T$ and $N$, respectively, along $\Gamma$. The geodesic curvature $k$ of $\Gamma$ is then defined by $s(T, T) = kN$. If we let $\tilde{T} = e^{-\psi} T$, $\tilde{N} = e^{-\psi} N$, then (3.2b) with $X = Y = T$ gives $e^{2\psi k}\tilde{N} = kN - (N\phi)N$, or

$$e^{\psi \tilde{k}} = k - N\phi,$$

a well known relation. Since $\nabla_T T$ and $\nabla_T N$ vanish, (3.3) implies that

$$(\nabla_T s)(T, T) = \nabla^2_T (kN) = (Tk)N.$$
Furthermore, $\alpha(T, T, T) = 0$ from (3.5), and this with (3.4) and (3.7) yields

\[(3.8) \quad e^{2\alpha T}\hat{k} = Tk - B(\varphi)(T, N).\]

For Möbius metrics the derivatives of $k$ and $\hat{k}$ are therefore proportional. In particular, this is the case for the euclidean curvature of a curve, and its curvature in the spherical metric on $\mathbb{R}^2$ or in the hyperbolic metric of a disk or a half-plane. One nice consequence of this is that one may deduce the classical four-vertices theorem for a curve in the hyperbolic plane or on the sphere from the usual version of the theorem for euclidean geometry.

In the classical case of analytic functions, (3.8) expresses the relationship between the derivative of the euclidean curvature of a curve and that of its image under a conformal mapping. This formula, together with the transformation formula (2.1), found important applications in a paper of Gehring [G]. Here, we note that from (3.8) it follows easily that a conformal diffeomorphism between surfaces maps (all) curves of constant geodesic curvature to curves of constant geodesic curvature if and only if it is a Möbius transformation.

Much of the preceding discussion carries over to umbilic points of a submanifold $P$ of $M$, the dimensions of $P$ and $M$ being arbitrary. Recall that a point $x \in P$ is called umbilic (with respect to the metric $g$) if there exists $A \in N_xP$ such that

\[(3.9) \quad s(X, Y) = \langle X, Y \rangle A\]

for all $X, Y \in T_xP$. A curve in $M$ is always totally umbilic, and while totally umbilic submanifolds of higher dimension are generally rare, given any subspace $E \subset T_xM$ and any $A \in E^\perp$, one can always find a submanifold $P$ through $x$ with $T_xP = E$ for which (3.9) holds. (Exponentiate a sphere in $T_xM$.)

At an umbilic point $x$ of $P$, (3.9) implies that

\[(3.10) \quad s((\grad \varphi)^T, X) = (X \varphi) A\]

for any smooth function $\varphi$ on $M$, and from (3.2b)

\[(3.11) \quad s(X, Y) = \langle X, Y \rangle \{A - (\grad \varphi)^N\}\]

for the metric $\hat{g} = e^{2\alpha}g$. That is, $x$ is also an umbilic point for $P$ with respect to the metric $\hat{g}$. Furthermore, using (3.10) in (3.5), we see that $\alpha(X, Y, Z) = 0$ for $X, Y, Z \in T_xP$, and at $x$, (3.4) becomes simply

\[(3.12) \quad (\nabla_g \varphi)(X, Y) = (\nabla_g s)(X, Y) - \langle X, Y \rangle \{M(\varphi)Z\}^N.\]

To cast this in another form, recall that the mean curvature vector $S$ along $P$ is defined to be

\[S = \frac{1}{p} \tr(s), \quad p = \dim P.\]
Then (3.2b) yields
\[ e^{2\varphi}S = S - (\text{grad } \varphi)^N \] (3.13)

at any point, while at umbilic point (3.12), (3.3) and (3.2b) lead to the relation
\[ e^{2\varphi}\nabla^2_Z S = \nabla^2_Z S - \{M(\varphi)Z\}^N. \]

Now, in the case that \( P \) is a hypersurface it is more convenient to use the \textit{scalar second fundamental form} \( h \), defined by
\[ s(X, Y) = h(X, Y)N \]
where \( N \) is a unit field normal field along \( P \). In this situation, if we let
\[ \tilde{N} = e^{-\varphi}N, \quad \tilde{s}(X, Y) = \tilde{h}(X, Y)\tilde{N}, \]
then (3.4) will become
\[ e^{-\varphi}(\tilde{\nabla}_Z h)(X, Y) = (\nabla_Z h)(X, Y) + \tilde{h}(X, Y, Z) - \langle X, Y \rangle B(\varphi)(Z, N), \] (3.14)

where, with reference to (3.5), \( \tilde{h} \) is the same as \( h \) with \( s \) replaced by \( h \). The \textit{mean curvature scalar} \( H \), given either by \( S = H N \) or by \( H = (1/p)\text{tr}(h), p = \dim P \), satisfies, along \( P \),
\[ e^{\varphi}\hat{H} = H - N\varphi, \]
by (3.13), and at an umbilic point, \( \tilde{h} = 0 \) in (3.14), leading to
\[ e^{\varphi}Z\hat{H} = ZH - B(\varphi)(Z, N) \] (3.15)
at that point. The relation holds along any totally umbilic submanifold and reduces, as it should, to (3.8) for a curve on a surface.

We can now establish the following theorem.

**Theorem 3.1.** \textit{If} \( \dim M \geq 3 \) \textit{and} \((M, g)\) \textit{is of constant curvature, then the M"obius and conformal groups coincide.}

This will also emerge as a corollary of a more general result in Section 4, Corollary 4.2, but the proof here makes use only of (3.15) and some elementary geometry of constant curvature manifolds.

**Proof of Theorem 3.1.** Let \( f : (M, g) \rightarrow (M, g) \) be conformal with \( f^*g = \hat{g} = e^{2\varphi}g \). We are to show that \( B_\varphi(\varphi) = 0 \). Let \( x \in M \) and let \( Z, N \in T_xM \) with \( \langle Z, N \rangle = 0 \), \( \|N\| = 1 \). Because \( M \) has constant curvature, we can find a totally umbilic hypersurface \( P \) through \( x \), \( \dim P \geq 2 \), with extensions \( Z, N \) tangent and normal to \( P \),
respectively. Furthermore, (see e.g., [O'N]), $P$ has constant mean curvature $H$, so that $ZH = 0.$ Now, umbilic points are preserved by conformal mappings, as follows from (3.11); hence, $f(P)$ is again totally umbilic and so also has constant mean curvature. Since $f: (M, \hat{g}) \to (M, g)$ is an isometry, we conclude that $\hat{H}$ is constant along $P$; hence, $ZH = 0$ and $B(\varphi)(Z, N) = 0$ by (3.15). Because the trace of $B(\varphi)$ vanishes and $Z$ and $N$ are in arbitrary orthogonal directions in $T_xM,$ this implies that $B(\varphi)(x) = 0$ and therefore that $B(\varphi) = 0$ identically.

4. The Schwarzian tensor and the Riemann curvature tensor. The Schwarzian tensor has a special meaning in dimension three or more; it tells how one of the natural components of the Riemann curvature tensor changes under a conformal change of metric. In this section we discuss these natural components and examine integrability conditions for the equation $B(\varphi) = 0.$

Let $E$ be a real vector space of dimension $n \geq 2$ with inner product $g = \langle \cdot, \cdot \rangle.$ A curvature tensor in $E$ is a bilinear mapping $R: E \times E \to \text{End}(E)$ such that

(i) $R(X, Y) = -R(Y, X),$

(ii) $\langle R(X, Y)Z, W \rangle = -\langle Z, R(X, Y)W \rangle,$ and

(iii) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0.$

The simplest examples are the tensors

$$kI_g: (X, Y, Z) \mapsto k\{\langle Y, Z \rangle X - \langle X, Z \rangle Y\}, \quad k \in \mathbb{R}.$$

These are characterized by the constancy of their sectional curvatures, every sectional curvature of $kI_g$ being equal to $k.$ Further examples are the tensors

$$R_\beta: (X, Y, Z) \mapsto \langle Y, Z \rangle \sum \beta(X, e_j)e_j - \langle X, Z \rangle \sum \beta(Y, e_j)e_j + \beta(Y, Z)X - \beta(X, Z)Y,$$

constructed from bilinear, symmetric, trace-free functions $\beta$ on $E \times E,$ where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $E.$ This construction produces a curvature tensor regardless of the trace of $\beta,$ but we use it only when the trace vanishes. Note that a conformal change of inner product does not affect the definition of $R_\beta.$ In dimension two, $R_\beta = 0$ for all $\beta.$

The Ricci contraction, defined by $\text{Ric}(R)(Y, Z) = \sum \langle R(e_j, Y)Z, e_j \rangle,$ maps the space $\mathcal{R}$ of curvature tensors into the space of bilinear, symmetric functions on $E \times E.$ Since

$$\text{Ric}(kI_g + R_\beta) = (n - 1)k + (n - 2)\beta,$$

this mapping is surjective when $n \geq 3.$ One can show that the dimension of $\mathcal{R}$ is equal to $n^2(n^2 - 1)/12.$ By counting dimensions, one then sees that

$$\text{(4.1) } \mathcal{R} = \{kI_g: k \in \mathbb{R}\} \oplus_{(a \geq 3)} \{R_\beta: \text{trace}(\beta) = 0\} \oplus_{(a \geq 4)} \{C \in \mathcal{R}: \text{Ric}(C) = 0\}.$$
The subscripts here indicate the dimensions in which the summands are nonzero; they are not logically necessary. Since a three-term decomposition might be confusing in dimension two, where \( \beta \) is ambiguous, we use it only when \( n \geq 3 \). When \( n = 3 \), the space \( \{ C \} \) of Weyl conformal tensors vanishes.

**Theorem 4.1.** Let \( (M, g) \) be a Riemannian manifold of dimension \( n \geq 2 \) and let \( \hat{g} = e^{2\phi} g \). If \( R = kI_g + R_g + C \) and \( \bar{R} = kI_{\hat{g}} + R_{\hat{g}} + \bar{C} \) are the Riemann curvature tensors of \( g \) and \( \hat{g} \), respectively, then

(a) \( \bar{k} = e^{-2\phi}(k - (2/n)\Delta \phi - ((n - 2)/n)\|{\text{grad}}\ \phi\|^2) \),
(b) \( \bar{\beta} = \beta - B(\phi) \quad (n \geq 3) \),
(c) \( \bar{C} = C \quad (n \geq 3) \),

where \( \Delta \phi \), \( \|{\text{grad}}\ \phi\| \), and \( B(\phi) \) are computed with respect to \( g \).

These formulas follow from equation (2.0) (see [E] p. 90, [Be] pp. 58–59).

An **Einstein manifold** is a Riemannian manifold in which the Ricci tensor is a constant multiple of the metric. Thus, a space of dimension three or more is Einsteinian if \( \beta \) vanishes and \( k \) is constant. When the space is connected, the latter condition is redundant by equation (4.2) below, and the Einstein condition can be written in the concise form \( \beta = 0 \). Theorem 4.1 therefore implies the following corollary.

**Corollary 4.2.** Let \( (M, g) \) be a connected Einstein manifold of dimension \( n \geq 3 \) and let \( \hat{g} = e^{2\phi} g \). Then \( (M, \hat{g}) \) is Einsteinian if and only if \( B(\phi) = 0 \). In particular, the conformal and Möbius groups of \( M \) coincide.

Theorem 4.1 is at least consistent with the main formula (3.4) from Section 3. Let \( P \) be a submanifold of the Riemannian manifold \( (M, g) \), and let \( s \) be its second fundamental form. The Codazzi equation asserts that the expression

\[
(V_s)(X, Y) - \frac{1}{3} \{R(Z, X)Y + R(Z, Y)X\}^N, \quad X, Y, Z \in T_xP,
\]

is symmetric in all three arguments. Suppose that \( \text{dim}(M) \geq 3 \) and that \( \hat{g} = e^{2\phi} g \).

By subtracting the Codazzi equations in \( g \) and \( \hat{g} \) and using Theorem 4.1 to evaluate \( \bar{R} - R \), one finds that the expression

\[
(V_s)(X, Y) - (V_s)(X, Y) + \langle X, Y \rangle \{B(\phi)(Z, e_j)e_j\}^N
\]

is also totally symmetric, as equation (3.4) says it should be.

It is important to see what the second Bianchi identity

\[
(V_{\nu}R)(X, Y) + (V_{\nu}R)(Y, V) + (V_{\nu}R)(V, X) = 0
\]

says about the covariant derivatives of \( k \), \( \beta \), and \( C \). This identity is trivial in dimension two since the left side is an alternating function of \( (V, X, Y) \). In dimension
n \geq 3$, let

\[
(\text{div } \beta)(X) = \sum (\nabla_{e_j} \beta)(X, e_j),
\]

\[
(\text{div } C)(X, Y; Z) = \sum \langle (\nabla_{e_j} C)(X, Y)Z, e_j \rangle,
\]

\[
C'(X, Y; Z) = (\nabla_X \beta)(Y, Z) - (\nabla_Y \beta)(X, Z) + \frac{1}{n-1} \{ (\text{div } \beta)(X) \langle Y, Z \rangle - (\text{div } \beta)(Y) \langle X, Z \rangle \}.
\]

Contracting the Bianchi identity twice and using the contracted equations to simplify the earlier ones, one obtains the identities

\begin{align*}
(4.2) & \quad 2(\text{div } \beta)(X) = (n-1)X_k, \\
(4.3) & \quad (\text{div } C)(X, Y; Z) = (n-3)C'(X, Y; Z), \\
(4.4) & \quad 0 = \mathcal{P}\{(n-3)\langle (\nabla_Y C)(X, Y)Z, W \rangle \\
& \quad + \langle V, Z \rangle (\text{div } C)(X, Y; W) - \langle V, W \rangle (\text{div } C)(X, Y; Z)\},
\end{align*}

where $\mathcal{P}$ means to sum over cyclic permutations of $(V, X, Y)$. By equation (4.2), the last terms in the definition of $C'$ are equal to $(1/2)\{(X_k)(Y, Z) - (Y_k)(X, Z)\}$. Equations (4.3) and (4.4) are of course trivial in dimension three.

In a precise sense, equation (4.1) gives the natural components of the Riemann curvature tensor, and equations (4.2)–(4.4) are the natural components of the second Bianchi identity. Again, let $E$ be a real inner product space of dimension $n \geq 2$. One might ask which pairs $(R, S)$ of tensors in $E$ can be realized as the Riemann curvature tensor and its covariant derivative at some point in some Riemannian manifold. So far, we have dealt with necessary conditions: $R$ must be a curvature tensor, and $S$ must be an element of $\text{Lin}(E, \mathcal{R}) = E^* \otimes \mathcal{R}$ such that

\[
0 = \mathcal{P}(S)(V; X, Y) = S(V; X, Y) + S(X; Y, V) + S(Y; V, X).
\]

It can be shown that these conditions are also sufficient. Therefore, any general properties of the Riemann curvature tensor and its covariant derivative follow from those already discussed. Clearly, $\mathcal{R}$ is invariant under the natural representation of the orthogonal group $O(E)$; in fact, each of the summands in the decomposition (4.1) is invariant. Using the tools of representation theory, one can show that these constituent $O(E)$-submodules are irreducible. Thus, equation (4.1) gives the complete decomposition of $\mathcal{R}$. Similarly, equations (4.2)–(4.4) reflect the decomposition of the $O(E)$-module $\mathcal{F} = \mathcal{P}(E^* \otimes \mathcal{R})$ into its irreducible components. In dimension
$n \geq 3$ this decomposition has the form

$$\mathcal{F} = \mathcal{F}_1 \oplus_{(n \geq 4)} \mathcal{F}_2 \oplus_{(n \geq 5)} \mathcal{F}_3$$

with

$$\mathcal{P}(E^* \otimes \{kI_g\}) = \mathcal{F}_1, \quad \mathcal{P}(E^* \otimes \{R_g\}) = \mathcal{F}_1 \oplus \mathcal{F}_2, \quad \mathcal{P}(E^* \otimes \{C\}) = \mathcal{F}_2 \oplus \mathcal{F}_3.$$

Equations (4.2)–(4.4) are the statements that the components of $\mathcal{P}(\nabla R)$ vanish. It is somewhat surprising to find that when $n = 4$, equation (4.4) provides no information; it holds for every element of $E^* \otimes \{C\}$.

The results of this section can be useful in a direct approach to the equation $B(\varphi) = p$ in a Riemannian manifold. With the usual substitution $u = e^{-\varphi}$, this becomes the linear equation

$$(4.5) \quad \text{Hess}(u) + up = \lambda g, \quad \lambda = \Delta u/n.$$ 

Let $W = \text{grad } u$. Using equation (4.5) to evaluate the Ricci tensor on $(X, W)$, one obtains the first-order system

$$0 = Xu - \langle X, W \rangle,$$

$$(4.6) \quad 0 = \nabla_X W + u \sum p(X, e_j)e_j - \lambda X,$$

$$0 = X\lambda + k\langle X, W \rangle + \left(\frac{n - 2}{n - 1}\right)\beta(X, W) + \frac{1}{n - 1} (p(X, W) + u(\text{div } p)(X)),$$

the term involving $\beta$ being interpreted as zero when $n = 2$. This system is clearly equivalent to equation (4.5); if $(u, W, \lambda)$ solves the system, then indeed $W = \text{grad } u$ and $\lambda = \Delta u/n$, and $u$ solves the equation.

The right sides of (4.6) define a connection in the vector bundle $\mathbb{R} \times TM \times \mathbb{R}$ for which the (local) parallel sections are the solutions of the system. A natural question is: When is this connection flat? Equivalently, when does every initial-value problem for the system admit a local solution? The answer is easily computed. In dimension two, flatness obtains precisely when a kind of Cauchy-Riemann equation holds:

$$(4.7) \quad (\nabla_X p)(Y, Z) - (\nabla_Y p)(X, Z) + \frac{1}{2} \{Xk, \langle Y, Z \rangle - \langle Yk, \langle X, Z \rangle \} = 0.$$
This, in turn, is equivalent to the simpler equation \((\text{div } p) (X) = Xk/2\). In dimension \(n \geq 3\) one finds, using equations (4.2)–(4.4), that flatness obtains when

\[(4.8) \quad p = \beta, \quad C' = 0, \quad \text{and } C = 0.\]

In Section 2 we introduced the notion of full integrability and showed that the equation \(B(\phi) = 0\) is fully integrable if and only if \(M\) has constant curvature. It is not difficult, particularly when \(p = 0\), to see that full integrability is equivalent to flatness of the connection. Equations (4.7) and (4.8) therefore provide a second proof of Theorem 2.3.

The bundle can surely admit a nonzero parallel section without the connection being flat. However, it is difficult to identify conditions other than flatness which guarantee that such sections exist. For example, consider the equation \(B(\phi) = \beta\) in dimension \(n \geq 3\). This is the problem of finding a conformal Einstein metric. To start, one might look for points in the bundle at which the connection is flat—that is, points which are annihilated by all the curvature tensors \(F(X, Y)\) of the connection—since a parallel section must map to such points. In the present example, one finds that the connection is flat at a point \((u, W, \lambda)\) in the fiber over \(x \in M\) if and only if

\[(4.9) \quad \langle C(X, Y) Z, W \rangle = uC'(X, Y; Z) \quad \text{for all } X, Y, Z \in T_x M.\]

This leads to necessary conditions for the existence of local solutions to \(B(\phi) = \beta\). At each point of \(M\), for example, equation (4.9) must admit a solution with \(u > 0\). Sufficient conditions, however, are an awkward matter.

When \(C\) vanishes, the equation \(B(\phi) = \beta\) becomes tamer. By (4.9), a solution can exist only if \(C'\) vanishes as well, in which case the connection is flat everywhere. Of course, \(C\) always vanishes in dimension three, and equation (4.3) shows that \(C'\) vanishes whenever \(C\) does in dimension \(n \geq 4\). Thus we obtain the following theorem.

**Theorem** (Weyl, Schouten) [E, p. 92]. *For a Riemannian manifold of dimension \(n \geq 4\) to be locally conformal to a space of constant curvature, it is necessary and sufficient that the Weyl conformal tensor vanish. In dimension three it is necessary and sufficient that the tensor \(C'\) vanish. When these conditions hold, the value of the sectional curvature and initial values of the conformal factor \(e^\psi\) and its gradient may be prescribed arbitrarily.*

### 5. The Möbius equation and warped products

The Möbius equation \(B(\phi) = 0\) was first studied by Brinkmann in his work on conformal transformations between Einstein manifolds [Br]. Brinkmann showed that near a regular point of a solution \(\phi\) the metric can be expressed in the form

\[
\frac{ds^2}{dx_1^2 + f(x_1)^2} \sum_{i,j=2}^n g_{ij}(x_2, \ldots, x_n) \, dx_i \, dx_j
\]
with \( \varphi \) being a function of \( x_1 \). This result, which holds for the Möbius equation in general, has been proved many times since, and indeed we continue the tradition. The purpose of this section, however, is broader. Consider a Riemannian manifold \((M, g)\); in our discussions, \( M \) is either a complete space or a small open set in a larger space. Recall that \( \mathcal{H}(M) \) denotes the space of solutions of the equation

\[
(5.1) \quad \text{Hess } u = \lambda_u g, \quad \lambda_u = \Delta u/\text{dim}(M),
\]

which is obtained from the Möbius equation by the substitution \( u = e^{-\varphi} \). We show how one can use the full space \( \mathcal{H}(M) \) to express the metric \( g \) as a special kind of warped product; conversely, we determine \( \mathcal{H}(M) \) when the metric is of that form. As an application, we then classify the complete Einsteinian manifolds which admit a nonhomothetic conformal metric.

Often \( \mathcal{H}(M) = \mathbb{R} \); that is, the only solutions of equation (5.1) are the constant functions. At the other extreme are spaces of constant curvature, where Theorem 2.3 asserts that there are many solutions, at least locally.

**Definition.** For a connected Riemannian manifold \( M \) and a real number \( K \), \( \mathcal{H}_K(M) \) is the space of functions \( u \in \mathcal{H}(M) \) such that \( \lambda_u + Ku \) is constant.

If \( u, v \in \mathcal{H}_K(M) \), then the expression

\[
(5.2) \quad \langle u, v \rangle = u \lambda_v + v \lambda_u - \langle \text{grad } u, \text{grad } v \rangle + Ku
\]

is also constant, as one sees by differentiating. This scalar-valued product will be very useful in what follows. Note that \( \lambda_u + Ku \) can be written as \( \langle u, 1 \rangle \).

**Theorem 5.1.** (a) If \( I \) is an interval, then \( \mathcal{H}_K(I) \) is three-dimensional, and \( \langle , \rangle \) is a nondegenerate product of signature \((+, -, -)\).

(b) Let \((M, g)\) be a connected space of constant curvature \( K \) and dimension \( n \geq 2 \). Then \( \mathcal{H}_K(M) = \mathcal{H}(M) \). Whenever \( u \in \mathcal{H}(M) \) is positive, \( \langle u, u \rangle \) is the constant curvature of the metric \( u^{-2}g \). If \( M \) is also simply connected, then \( \mathcal{H}(M) \) has dimension \( n + 2 \), and \( \langle , \rangle \) is a nondegenerate product of signature \((+, -, \ldots, -)\).

**Proof.** In an interval, \( \mathcal{H}_K \) consists of the solutions of \( u'' + Ku' = 0 \) and so is three-dimensional. By explicit computation, one sees that the product has signature \((+, -, -)\) in this case.

Part (b) follows from the results of Section 4. Here, we are interested in solutions of the system (4.6), with \( p = 0 \), in a space of constant curvature \( K \). The first and third equations of (4.6) show that \( \lambda_u + Ku \) is constant for any solution \( u \); where \( u \) is positive, Theorem 4.1 shows that the metric \( u^{-2}g \) has constant curvature \( \langle u, u \rangle \).

We have seen that the connection corresponding to the system (4.6) is flat in this case. When \( M \) is simply connected, then, there is exactly one solution for each initial value \((u_0, w_0, \lambda_0)\) in a specified fiber. In particular, \( \mathcal{H}(M) \) has dimension \( n + 2 \), and from the expression \( \langle u, u \rangle = 2u \lambda_u - \| \text{grad } u \|^2 + Ku^2 \), one sees that the product has signature \((+, -, \ldots, -)\).
One can also derive part (b) from the results of Section 2. If $M$ has constant curvature $K$, then it is locally isometric to a Möbius metric in $S^n$ which in stereographic coordinates has the form

$$(a|x|^2 + (b \cdot x) + c)^{-2} \text{euc}, \quad a, c \in \mathbb{R}, \quad b \in \mathbb{R}^n, \quad 4ac - |b|^2 = K;$$

if $M$ is simply connected, it admits a global isometric development into this model. By equation (2.11), the general solution to equation (5.1) in the model is

$$(5.3) \quad u(x) = \frac{A|x|^2 + (B \cdot x) + C}{a|x|^2 + (b \cdot x) + c}, \quad A, C \in \mathbb{R}, \quad B \in \mathbb{R}^n.$$ 

The assertions of part (b) follow immediately. In terms of these coefficients, $\langle u, u \rangle_K$ is simply $4AC - |B|^2$.

As an application of this product, we derive the law of cosines in hyperbolic, Euclidean, and spherical geometry. Let $M$ be a simply connected space of constant curvature $K \in \{-1, 0, 1\}$. For $p \in M$, let $u_p$ denote the element $u \in \mathcal{H}(M)$ which vanishes to first order at $p$ with $\lambda_u(p) = 1$. If $u(t)$ denotes the value of $u_p$ along a unit-speed geodesic emanating from $p$, then $u'' + Ku = \langle u, 1 \rangle_K = 1$, and $u(0) = u'(0) = 0$. It follows that

$$u(t) = \begin{cases} \cosh t - 1 & \text{if } K = -1, \\ t^2/2 & \text{if } K = 0, \\ 1 - \cos t & \text{if } K = 1. \end{cases}$$

Now consider a geodesic triangle $ABC$ in $M$. Evaluating $\langle u_A, u_B \rangle_K$ at $B$, where three of the four terms in equation (5.2) vanish, one finds that this product is equal to $u_A(B)$. Evaluating it at $C$ and equating the results, one obtains the law of cosines

$$K = -1: \quad \cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma,$$

$$K = 0: \quad c^2 = a^2 + b^2 - 2ab \cos \gamma,$$

$$K = 1: \quad \cos c = \cos a \cos b + \sin a \sin b \cos \gamma.$$ 

Here $a, b, c$ are the sides opposite $A, B, C$, respectively, and $\gamma$ is the angle at $C$.

The metrics which appear in connection with the Möbius equation are warped products. Let $Q$ and $P$ be Riemannian manifolds and let $f$ be a positive function on $Q$. The warped product $Q \times_f P$ is the differentiable manifold $Q \times P$ with Riemannian metric

$$ds^2 = (ds^Q)^2 + \tilde{f}^2(ds^P)^2, \quad \tilde{f}(q, p) = f(q).$$

Because $\tilde{f}$ is emphasized in what follows, we note that $\text{grad} \tilde{f}$, as computed in the warped product metric, is simply the horizontal vector field which projects to $\text{grad}^Q f$. Indeed, the same holds for any function on $Q$ and its lift to $Q \times_f P$. 
To describe the Riemannian connection in a warped product $M = Q \times_f P$, it is sufficient to consider vector fields which are horizontal or vertical lifts, in the obvious sense, of vector fields on $Q$ or $P$. To simplify notation, we use the same symbol for the lift as for the vector field from which it derives. The covariant derivative $\nabla$ in $M$ is then characterized by the conditions that, for all lifted horizontal vector fields $A$, $B$ and all lifted vertical vector fields $X$, $Y$ on $M$,
\[
\nabla_A B = \nabla^Q_A B,
\]
(5.4)
\[
\nabla_A X = \nabla_X A = A(\log \tilde{f}) X,
\]
\[
\nabla_X Y = \nabla^P_X Y - \langle X, Y \rangle \text{grad}(\log \tilde{f}).
\]

Here $\langle \cdot, \cdot \rangle$ denotes the metric on $M$. From these equations, one sees that the horizontal slices $Q \times \{p\}$ are totally geodesic and that the vertical slices $\{q\} \times P$ are totally umbilic. A further computation shows that $M$ has constant curvature $K$ precisely when

(i) $Q$ is either one-dimensional or a space of constant curvature $K$,

(ii) $f \in \mathcal{H}_K(Q)$ with $\langle \langle f, 1 \rangle \rangle_K = 0$, and

(iii) $P$ is either one-dimensional or a space of constant curvature $L = -\langle \langle f, f \rangle \rangle_K$.

See O'Neill's book [O\'N] for a more complete discussion.

A similar construction produces polar and spherical warped products. Let $f$ be a function on the interval $[0, T)$ which vanishes at zero and is positive elsewhere, such that $f'(0) = 1$ and the odd extension of $f$ is smooth. We define Polar($f$) to be the Riemannian manifold whose underlying space is the ball $|x| < T$ in $\mathbb{R}^n$ and whose metric is given in polar coordinates by
\[
ds^2 = dt^2 + \tilde{f}^2 d\theta^2, \quad \tilde{f}(t, \theta) = f(t).
\]

Because $f'(0) = 1$, the metric is indeed smooth through the origin. This construction amounts to collapsing the boundary of $[0, T) \times S^{n-1}$ to a point, using polar coordinates to define the differential structure near that point, and imposing the metric (5.5). If $f$ is defined on $[0, T]$, one can similarly collapse each boundary component of $[0, T] \times S^{n-1}$ and obtain a Riemannian manifold Sphere($f$) which is diffeomorphic to $S^n$; here, $f$ must also vanish at $T$, with $f'(T) = -1$, and the odd, $2T$-periodic extension of $f$ must be smooth, so that the metric is smooth through the second pole. The observations above show that Polar($f$) and Sphere($f$) have constant curvature $K$ away from the poles, and hence everywhere, precisely when $f'' + Kf = 0$.

We now show how the existence of nonconstant solutions to equation (5.1) leads to local warped-product structures.
THEOREM 5.2. Let $M$ be a Riemannian manifold of dimension $n \geq 2$.

(a) Let $x$ be a regular point of $u \in \mathcal{U}(M)$. Then near $x$ the metric can be expressed as a warped product $I \times_f P$, where $I$ is an interval, the slices $\{t\} \times P$ are the level sets of $u$, and $\tilde{f}$ is a constant multiple of $\| \text{grad } u \|$.

(b) Let $x$ be a regular point of $u = (u_1, \ldots, u_m)$, where $u_1, \ldots, u_m \in \mathcal{U}(M)$ and $2 \leq m < n$. Then near $x$ the metric can be expressed as a warped product $Q \times_f P$, where $Q$ is a space of constant curvature $K \in \mathbb{R}$, the slices $\{q\} \times P$ are the level sets of $u$, and $\tilde{f}$ is a constant multiple of $\| \text{grad } u_1 \wedge \cdots \wedge \text{grad } u_m \|$. Furthermore, $f \in \mathcal{U}(Q)$ with $\langle \tilde{f}, 1 \rangle_K = 0$.

(c) Let $x$ be a regular point of $u = (u_1, \ldots, u_n)$, where $u_1, \ldots, u_n \in \mathcal{U}(M)$. Then $M$ has constant curvature near $x$.

(d) Let $x$ be a critical point of $u \in \mathcal{U}(M)$, where $u$ is constant in no neighborhood of $x$. Then $x$ is a nondegenerate local minimum or maximum, and the level sets of $u$ nearby are the metric spheres about $x$. Near $x$, the metric can be expressed as a polar warped product $\text{Polar}(f)$ centered at $x$, with $\tilde{f} = \| \text{grad } u \|/|\lambda_u(x)|$.

Part (a) is Brinkmann's result [Br]. It is also proved by Yano [Yn], Tashiro [T2], Kulkarni [K1], and Yau [Y]. We supply a proof, in part for the sake of completeness, but also because we must cover the same ground to prove part (b). Part (c) has already been established in Theorem 2.3, and we use this result in proving (b). The final assertion (d) is proved in [T2], [K1], and [Y]; for completeness, we also supply a proof.

Proof. We prove parts (a) and (b) together. Here, $u_1, \ldots, u_n \in \mathcal{U}(M)$, $1 \leq m < n$, and the gradients $G_j = \text{grad } u_j$ are linearly independent at $x$. If $\lambda_j = \lambda_{u_j}$, the equation

$$[G_i, G_j] = \text{grad } G_i \text{ grad } G_j = \lambda_j G_i - \lambda_i G_j$$

shows that the field of $m$-planes spanned by these gradients integrates to a foliation in a neighborhood of $x$. Let $Q$ be a small, connected neighborhood of $x$ in the leaf through $x$, and let $P$ be a small, connected neighborhood of $x$ in the level set $u = u(x)$. We parametrize a neighborhood of $x$ by $Q \times P$ in the obvious way, using the leaves of the foliation and the level sets of $u$ to define the projections.

Consider $Q \times P$ with the metric obtained from $M$. The horizontal slices $Q \times \{p\}$ and the vertical slices $\{q\} \times P$ are of course orthogonal in this metric, and $u$ is constant on each fiber $\{q\} \times P$. We show that

(i) $\langle G_i, G_j \rangle$ is constant on each fiber $\{q\} \times P$.

(ii) $G_j$ is the horizontal lift of its restriction to $Q$.

(iii) If $X$ is the vertical lift of a vector field on $P$, then $\| X(q, p) \| = f(q, p) \| X(p) \|$, where

$$f(q, p) = \frac{\| (G_1 \wedge \cdots \wedge G_m)(q, p) \|}{\| (G_1 \wedge \cdots \wedge G_m)(p) \|}.$$ 

 Assertion (i) follows from the fact that $X(\langle G_i, G_j \rangle) = \langle \lambda_i X, G_j \rangle + \langle G_i, \lambda_j X \rangle = 0$ for any vertical vector $X$. To establish (ii), let $\bar{G}_j$ be the horizontal lift of the restriction
of \( G_j \) to \( Q \). As the fibers of our product are the level sets of \( u \), this is the unique horizontal vector field such that \( \tilde{G}_j(u)(q, p) = G_j(u)(q) \). But \( G_j \) itself has this property, since \( G_j(u_t) = \langle G_j, G_j \rangle \) is constant on each fiber. Therefore, \( G_j = \tilde{G}_j \). Finally, let \( X \) be a lifted vertical vector field as in assertion (iii). We must show that \( \|X\|/\|G_1 \wedge \cdots \wedge G_m\| \) is constant on each leaf \( Q \times \{ p \} \), or equivalently that the derivative of this expression in the direction of any of the vectors \( G_j \) is zero. Consider the covariant derivative of \( G_1 \wedge \cdots \wedge G_m \) in the direction of \( G_j \). Using the multilinearity of the wedge product, one obtains \( m \) terms, all but one of which contain two factors of \( G_i \) and hence vanish. Therefore this covariant derivative is equal to 

\[
\lambda_j (G_1 \wedge \cdots \wedge G_m) = \lambda_j \|G_1 \wedge \cdots \wedge G_m\|.
\]

This follows from \( \nabla_{G_j} X = \nabla_{\tilde{G}_j} G_j = \lambda_j X \), we also have that \( G_j(\|X\|) = \lambda_j \|X\| \). These results imply that the derivative of \( \|X\|/\|G_1 \wedge \cdots \wedge G_m\| \) in the direction of \( G_j \) is zero, as desired.

By (i) and (ii), \( \tilde{f} \) is constant on each fiber \( \{q\} \times P \). If \( f \) denotes the restriction of \( \tilde{f} \) to \( Q \), then the three assertions show that our metric is the warped product \( Q \times f \). This establishes part (a) of Theorem 5.2. For part (b), suppose that \( m \geq 2 \). Since \( Q \) is totally geodesic, the Hessian operator in \( Q \) is just the restriction of the Hessian in \( M \). Therefore the restriction of \( u_j \) to \( Q \) is an element of \( \Psi (Q) \). By Theorem 2.3, \( Q \) has constant curvature \( K \) for some \( K \). Using equation (5.6) and the fact that \( \lambda_j + ku_j \) is constant on \( Q \), we find that

\[
G_j(G_j f) - (\nabla^2_{G_j} G_j)(f) = G_j(\lambda_j f) - \lambda_j G_j(f) = f G_j(\lambda_j) = -K f \langle G_j, G_j \rangle.
\]

Therefore, Hess\(^Q(\tilde{f}) + K f^Q = 0 \), which is to say that \( f \in \Psi (Q) \) with \( \langle \tilde{f}, 1 \rangle^Q = 0 \).

Finally, suppose that \( x \) is a critical point of \( u \in \Psi (M) \) but that \( u \) is not constant in any neighborhood of \( x \). As observed in Section 4, \( \lambda_u \) cannot vanish at \( x \), for this would imply that \( u \) was constant on the component of \( M \) containing \( x \). It follows that \( x \) is a nondegenerate local minimum or maximum, depending on the sign of \( \lambda_u(x) \). To be definite, we assume that \( \lambda_u(x) < 0 \). Then every integral curve of the vector field \( G = \nabla u \) starting near \( x \) converges to \( x \). Since \( \nabla^G G = \lambda_u G \), such a curve is a reparametrized geodesic. It follows that the level sets of \( u \) nearby are the metric spheres about \( x \).

Let \( u(t) \) be the value of \( u \) on the metric sphere of radius \( t \), where \( t \) lies in an appropriate interval \([0, T)\). Consider polar exponential coordinates \( (t, \theta) \) near \( x \), where \( t \in [0, T) \) and \( \theta \) is in the unit sphere \( S \subseteq T_x M \). In these coordinates, the metric has the form \( ds^2 = dt^2 + g_r(d\theta, d\theta) \) for some metrics \( g_r \) on \( S \). By part (a) of the present theorem,

\[
g_r = \left( \frac{u'(t)}{u(r)} \right)^2 g_r \quad \text{when} \quad t, r \in (0, T).
\]

Now fix \( t \) and let \( r \to 0 \). In polar exponential coordinates, \( g_r/r^2 \) always tends to the standard round metric. Since \( u'(r)/r \) tends to \( u''(0) = \lambda_u(x) \), it follows that \( g_r \).
equals \((u'(t)/\lambda_n(x))^2\) times the round metric. Thus we have a polar warped product \(\text{Polar}(f)\), where \(\tilde{f} = \|\nabla u\|/\lambda_n(x)\). This completes the proof of Theorem 5.2.

The next result is a kind of converse.

**Theorem 5.3.** Let \(M = Q \times f P\), where \(Q\) and \(P\) are connected Riemannian manifolds of dimension at least one.

(a) Suppose that \(\text{Hess}^Q(f)/f\) is not a constant multiple of \(g^Q\). Then \(\mathcal{U}(M)\) consists of the functions \(u(q, p) = h(q)\) such that \(h \in \mathcal{U}(Q)\) and \(f h = \langle \text{grad}^Q h, \text{grad}^Q f \rangle\).

(b) Suppose that \(\text{Hess}(f) + Kf = 0\), where \(K\) is constant. Let \(L = -\langle f, f \rangle_K\). Then \(\mathcal{U}(M)\) consists of the functions \(u(q, p) = h(q) + f(q)v(p)\) such that \(h \in \mathcal{U}_k(Q), v \in \mathcal{U}_L(P)\), and \(\langle f, u \rangle = \langle f, f \rangle_K\). In this case, \(\mathcal{U}(M) = \mathcal{U}_k(M)\), \(\langle f, u \rangle = \langle f, f \rangle_K\), and \(\langle u, u \rangle = \langle f, f \rangle_K + \langle f, v \rangle\).

At the beginning of this section, we said that the full space \(\mathcal{U}(M)\) could be used to reduce the metric to a special form and that one could determine \(\mathcal{U}(M)\) for a metric of that form. Theorems 5.2 and 5.3 give us the tools with which to fulfill that promise at the local level or, more precisely, at the level of germs of solutions to equation (5.1) and germs of Riemannian metrics. Theorem 5.4 below provides this at the global level. Since we do not need the local version, and since a correct presentation would be somewhat tedious, we leave this point to the reader.

**Proof of Theorem 5.3.** Let \(u \in \mathcal{U}(M)\), where \(M = Q \times f P\). If \(A\) is any lifted horizontal vector field on \(M\) and \(X\) is any lifted vertical vector field, equations (5.4) show that

\[
A(Xu) = \text{Hess}(u)(A, X) + (\nabla_A X)(u) = 0 + A(\log \tilde{f})(Xu),
\]

and hence \((Xu)/\tilde{f} = 0\). Therefore, \((Xu)/\tilde{f}\) is constant on the horizontal slices \(Q \times \{p\}\). This implies that \(u\) has the form \(u(q, p) = h(q) + f(q)v(p)\).

For any function \(u\) of this form,

\[
(\text{grad} u)(q, p) = \text{grad}^Q(h + v(p)f)(q) + \frac{1}{f(q)}(\text{grad}^P v)(p).
\]

If \(A\) is a horizontal vector and \(X\) is a vertical vector then, by equations (5.4),

\[
\nabla_A(\text{grad} u)_{(q, p)} = \nabla^Q_A \text{grad}^Q(h + v(p)f)_{(q)},
\]

\[
\nabla_X(\text{grad} u)_{(q, p)} = \frac{1}{f(q)}\{\langle \text{grad}^Q f, \text{grad}^Q(h + v(p)f) \rangle_q \cdot X + \nabla^P_X \text{grad}^P v \}_{(p)}.
\]

For such a function to be in \(\mathcal{U}(M)\), it is therefore necessary and sufficient that:

1. \(h + v(p)f \in \mathcal{U}(Q)\) for all \(p \in P\),
2. \(v \in \mathcal{U}(P)\), and
3. \(\lambda_n(h + v(p)f)(q) = \frac{1}{f(q)}\{\langle \text{grad}^Q f, \text{grad}^Q(h + v(p)f) \rangle(q) + \lambda_n(p)\} \text{ for all } (q, p) \in M.\)
Assume the hypotheses of part (a). Consider a function \( u(q, p) = h(q) + f(q)v(p) \) in \( \mathcal{U}(M) \), and for purposes of contradiction suppose that \( v \) is not constant; say, \( v(p_1) \neq v(p_2) \). By condition (1), both \( h \) and \( f \) are in \( \mathcal{U}(Q) \). Setting \( p = p_1 \) and then \( p = p_2 \) in condition (3) and subtracting, one finds that the function \( \frac{\lambda_f f - \|\text{grad}^0 f\|^2}{(v(p_1) - v(p_2))} \) on \( Q \) has the constant value \( (\lambda_u(p_1) - \lambda_u(p_2))/(v(p_1) - v(p_2)) \).

Therefore,
\[
0 = A(\lambda_f f - \|\text{grad}^0 f\|^2) = fA(\lambda_f) - \lambda_f A(f) = f^2 A(\lambda_f/f) \quad \text{for all } A \in TQ.
\]

But by hypothesis \( \lambda_f/f \) is not constant. The contradiction shows that \( v \) must be constant, and hence \( u \) is a function of \( q \) alone. In characterizing \( \mathcal{U}(M) \), then, it is sufficient to consider functions of the form \( u(q, p) = h(q) \).

Consider a function \( u(q, p) = h(q) + f(q)v(p) \) in \( \mathcal{U}(M) \). Since \( f \) is in \( \mathcal{U}(Q) \), condition (1) shows that \( h \) is, too. Condition (3) states that
\[
(5.7) \quad (\lambda_h - \langle \text{grad}^0 h, \text{grad}^0 f \rangle)(q) = (\lambda_v - (\angle f, f))(q) \quad \text{for all } (q, p) \in M.
\]

Clearly both sides of this equation must be constant. Therefore
\[
0 = A(\lambda_h - \langle \text{grad}^0 h, \text{grad}^0 f \rangle) = fA(\lambda_h) - \lambda_f A(f) = f^2 A(\lambda_f/f) \quad \text{for all } A \in TQ,
\]

and hence \( h \in \mathcal{U}_K(Q) \). Equation (5.7) then becomes the statement that \( v \in \mathcal{U}_L(P) \), where \( L = -\langle f, f \rangle_K \), with \( \langle v, 1 \rangle_L = \langle h, f \rangle_K \). Running the same argument in reverse, one sees that if \( h \in \mathcal{U}_K(Q) \) and \( v \in \mathcal{U}_L(P) \) are any functions satisfying this condition, then the corresponding function \( u \) is in \( \mathcal{U}(M) \). For the final assertions of part (b), note that
\[
(\lambda_u + Ku)(q, p) = \langle h + v(p)f, 1 \rangle_K = \langle h, 1 \rangle_K.
\]

Thus, \( \mathcal{U}(M) = \mathcal{U}_K(M) \), and \( \langle u, 1 \rangle_K = \langle h, 1 \rangle_K \). The expression for \( \langle u, u \rangle_K \) at \( (q, p) \) differs from that for \( \langle h + v(p)f, h + v(p)f \rangle_K \) at \( q \) only in that grad \( u \) has an additional vertical component \( (\text{grad}^P v)(p) \). It follows that
\[
\langle u, u \rangle_K = \langle h + v(p)f, h + v(p)f \rangle_K - \|\text{grad}^P v(p)\|^2
\]
\[
= \langle h, h \rangle_K + 2v(p) \langle v, 1 \rangle_L - v(p)^2 L - \|\text{grad}^P v(p)\|^2
\]
\[
= \langle h, h \rangle_K + \langle v, v \rangle_L.
\]

This completes the proof of Theorem 5.3.

For \( K < 0 \), let \( \mathcal{H}^m(K) \) denote the half-space \( x_m > 0 \) in \( \mathbb{R}^m \) with Riemannian metric \( ds = |dx|/(\sqrt{-K} x_m) \). This space has constant curvature \( K \) when \( m \geq 2 \).
THEOREM 5.4. Let $M$ be a complete, connected Riemannian manifold of dimension $n \geq 2$ such that $\mathcal{U}(M) \neq \mathbb{R}$. Then $M$ is isometric to one of the following:

I. a complete, simply connected space of constant curvature;

II. with $P$ complete and connected and $1 \leq m < n$

A. $\mathbb{R}^m \times P$, where $\mathcal{U}_0(P) = \mathbb{R}$,

B. $\mathbb{H}^m(K) \times_f P$, where $f(x) = 1/x^m$ and $\mathcal{U}_0(P) = \mathbb{R}$, or

C. $\mathbb{H}^m(K) \times_f P$, where $f(x) = (|x|^2 + 1)/(2x_m)$ and $\mathcal{U}_K(P) = \mathbb{R}$;

III. with $f''/f$ not constant

A. $\mathbb{R} \times_f P$, where $P$ is complete and connected,

B. Polar$(f)$, where $T = \infty$, or

C. Sphere$(f)$.

Equation (5.3) describes $\mathcal{U}$ explicitly for spaces of class I. For class II, Theorem 5.3(b) shows that $\mathcal{U}$ consists of the functions

$$u(x, p) = \begin{cases} B \cdot x + C & \text{(IIA)}, \\
(B \cdot x + C)/x_m & \text{(IIB)}, \\
(-C|x|^2 + B \cdot x + C)/x_m & \text{(IIC)},
\end{cases}$$

where $B \in \mathbb{R}^m$ and $C \in \mathbb{R}$. For class III, part (a) of the same theorem shows that $\mathcal{U}$ consists of the functions $u(t) = B \int_0^t f + C$. Thus the converse of Theorem 5.4 holds: if $M$ is isometric to one of these spaces, then $\mathcal{U}(M) \neq \mathbb{R}$.

We have chosen specific forms in class II in order to facilitate applications. For the proof, however, another description is convenient: class II consists of the warped products $Q \times_f P$, where

1. $Q$ is a line or a complete, simply connected space of constant curvature $K < 0$,
2. $f \in \mathcal{U}_K(Q)$ with $\langle f, 1 \rangle_K = 0$, and
3. $\mathcal{U}_K(P) = \mathbb{R}$, where $L = -\langle f, f \rangle_K$.

Clearly the forms presented in the theorem are of this kind. Conversely, consider a warped product satisfying conditions (1)–(3). If $K = 0$, one may take $Q$ to be $\mathbb{R}^m$, and the fact that $\langle f, 1 \rangle_K = 0$ implies that $f$ is an affine function. Since $f$ remains positive, it must be constant, and one can absorb this constant into the metric on $P$ to obtain the form IIA. If $K < 0$, one may take $Q$ to be $\mathbb{H}^m(K)$, and by pushing $f$ forward by an isometry of $Q$ and absorbing a constant factor into the metric of $P$ one can reduce $f$ to the form prescribed in IIB or IIC. Note that $\mathcal{U} = \mathcal{U}_K$ in class II. In IIA and IIB the product $\langle \cdot, \cdot \rangle_K$ is degenerate with signature $(0, -\ldots, -)$, and in IIC it is negative definite. This product is of no use in class III.

Proof of Theorem 5.4. Let $u$ be a nonconstant element of $\mathcal{U}(M)$. For a regular point $x$ of $u$, let $\Phi, x$ denote the maximally defined solution to the initial-value problem

$$\frac{d\Phi}{dt} = \frac{(\text{grad } u)(\Phi)}{\|(\text{grad } u)(\Phi)\|}, \quad \Phi_0 = x.$$
As we have seen, this solution is a geodesic. Furthermore, if $X \in T_x M$ is orthogonal to $(\text{grad } u)(x)$, then $d(\Phi_t)(x) \cdot X$ remains orthogonal to $\text{grad } u$, and

\begin{equation}
\left\|d(\Phi_t)(x) \cdot X\right\| = \frac{\left\|\text{grad } u(\Phi_t(x))\right\|}{\left\|\text{grad } u(x)\right\|} \left\|X\right\|,
\end{equation}

for as long as $\Phi_t x$ is defined. These assertions follow from part (a) of Theorem 5.2, or from direct verification that the quantities asserted to be equal satisfy a common initial-value problem.

Suppose that $u$ has at least two critical points. Let $x$ be one such point, and let $S$ be the unit sphere in $T_x M$. Since $u$ is not constant, $\lambda_u(x)$ is not zero. Replacing $u$ with its negative if necessary, we assume that $\lambda_u(x) > 0$. Thus $x$ is a nondegenerate local minimum. By Theorem 5.2(d), there exists $T' > 0$ such that, if $t \in [0, T')$, then (1) $u$ is constant on $\exp(tS)$, and (2) under the mapping $\theta \mapsto \exp(t\theta)$, the metric $g$ in $M$ pulls back to $f(t)^2$ times the standard spherical metric on $S$, where $f(t) = u'(t)/\lambda_u(x)$. Let $r$ be a positive number less than $T'$ and let $T$ be the distance from $x$ to the next nearest critical point of $u$. Using the identity $\exp(t\theta) = \Phi_{e^{t\theta}} \exp(r\theta)$, one sees from the comments of the previous paragraph that assertions (1) and (2) in fact hold whenever $t \in [0, T)$. Thus the exponential mapping is a local isometry from the metric $\text{Polar}(f)$ on the ball of radius $T$ in $T_x M$ into $M$. But the exponential mapping is injective on this ball; it is injective on each radial segment since $u$ is strictly increasing along the image, and it maps different radial segments to disjoint images since these images are integral curves of a vector field. Thus it is an isometry onto an open set in $M$.

Since $u'(T^-) = 0$, the diameter of $\exp(tS)$ tends to zero as $t \to T^-$. Therefore $\exp(TS)$ is a single critical point $z$. Consider a value $t$ only slightly less than $T$. Since $\exp(tS)$ is relatively open and closed in the metric sphere of radius $T - t$ about $z$, it is the entire sphere. Thus $\{z\} \cup \exp(B)$ is all of $M$, for it is both open and closed, and so $M$ is isometric to $\text{Sphere}(f)$. If $f''/f$ is not constant, then $M$ is of class IIIIC; if $f'' + Kf = 0$, then $M$ is of class I with constant curvature $K$.

By the same argument, one sees that if $u$ has just one critical point, then $M$ is isometric to a polar warped product $\text{Polar}(f)$, where $f$ is defined on $[0, \infty)$. If $f''/f$ is not constant, then $M$ is of class IIIB; if $f'' + Kf = 0$, then $M$ is of class I with constant curvature $K$.

Suppose, then, that $u$ has no critical point. Let $P$ be a component of a level set of $u$, and consider $\Phi$ as a mapping from $\mathbb{R} \times P$ into $M$. The comments of the first paragraph show that $u$ is constant on each set $\Phi([t] \times P)$ and that $\Phi^*g$ is the warped product metric $\mathbb{R} \times_f P$, where $f(t) = u(t)/u(0)$. In particular, $\Phi$ is a local diffeomorphism on this domain. It is also injective; it is injective on each vertical slice $\{t\} \times P$ since $\Phi_t$ is injective, and it clearly takes different vertical slices to disjoint images. Since $\Phi^*g$ is complete, the image is closed and open and hence is all of $M$. Therefore, $\Phi$ is a diffeomorphism onto $M$.

If $f''/f$ is not constant, then $M$ is of class IIIA. Now suppose that $f'' + Kf = 0$. Let $L = -\langle f, f' \rangle_K = Kf^2 + (f')^2$. Since $f$ does not vanish, one sees that $K$ and $L$...
are nonpositive. If $P$ is a line or a simply connected space of constant curvature $L$, then $M$ is of class I with constant curvature $K$. If $\mathcal{U}_l(P) = \mathbb{R}$, as happens when $P$ is a circle, then $M$ is of class II with $m = 1$. By induction on $n$, the only remaining possibility is that $n \geq 3$ and that $P$ is a warped product $Q \times f_1, P_1$ satisfying conditions (1)--(3) above, the constant being $L$ rather than $K$. In this case we write

$$M = Q \times_f P_1, \quad Q = \mathbb{R} \times_f Q_1, \quad F(t, x) = f(t)f_1(x).$$

Then $Q$ is a complete, simply connected space of constant curvature $K$ and, by Theorem 5.3(b), $\langle F, 1 \rangle_K = 0$. By the same theorem the value $L' = -\langle F, F \rangle_K$ is equal to $-\langle f_1, f_1 \rangle_L$, and so $\mathcal{U}_l(P_1) = \mathbb{R}$. Therefore $M$ is of class II. This completes the proof of Theorem 5.4.

We return now to the question considered by Brinkmann: Which Einstein manifolds admit a conformal but nonhomothetic Einsteinian metric? For this, the main step is to determine which of the metrics encountered in this section are Einsteinian. With the aid of equations (5.4), one finds that:

1. Metrics of constant curvature are so.
2. If $Q$ is an interval or space of constant curvature $K$, $f \in \mathcal{U}_K(Q)$, and $\langle f, 1 \rangle_K = 0$, then $Q \times_f P$ is Einsteinian if and only if $\text{Ric}^p = (\dim(P) - 1)Lg^p$, where $L = -\langle f, f \rangle_K$ (that is, if $P$ is one-dimensional or an Einstein space of constant $L$).
3. If $f''/f$ is not constant, then $I \times_f P$, Polar$(f)$, and Sphere$(f)$ are not Einsteinian.

On the level of germs, then, the Riemannian metrics which Brinkmann seeks are of two kinds: metrics of constant curvature and warped products $Q \times_f P$ as above in which $\mathcal{U}_l(P) = \mathbb{R}$. Yau mistakenly asserts that only the former occur [Y, Theorem 3]. His error has been noted by Barbance and Kerbrat [BK].

On the global level, Theorem 5.4 has the following corollary.

**Theorem 5.5.** The complete, connected Einstein manifolds of dimension $n \geq 3$ which admit a conformal but nonhomothetic Einsteinian metric are as follows:

(a) complete, simply connected spaces of constant curvature;
(b) $\mathbb{H}^n(K) \times_f P$, where $m \geq 1$, $f(x) = 1/x_m$, and $P$ is a complete, connected, Ricci flat space with $\mathcal{U}_0(P) = \mathbb{R}$.

This corrects Yau's assertion that the only such manifolds are the spheres [Y, Corollary 4.1]. Note that spaces of class IIA and IIC do not appear here, even though many are Einsteinian, since for such spaces the only positive elements of $\mathcal{U}$ are the positive constants.

In general, the conformal Einsteinian metric $\hat{g} = u^{-2}g$ will not be complete. In fact, suppose that $K \leq 0$ and that $u$ is not constant. Let $u(t) = u(y(t))$, where $y$ is a unit speed geodesic along which $u$ is not constant. Since $u'' + Ku$ is constant, one sees that at least one end of $y$ has finite length in $\hat{g}$, and so $\hat{g}$ is not complete. This proves the following theorem.
THEOREM 5.6. Let $M$ be a complete, connected Einstein manifold which admits a complete and conformal, but not homothetic, Einsteinian metric. Then $M$ is a round sphere.

Tashiro announces this theorem, without proof, as a consequence of another result which is in fact incorrect [T1, Theorem 3]. Kulkarni proves the theorem [K1]. Both papers, however, appear after a stronger theorem of Nagano [Na]. Using a result of Tanaka [Tk], Nagano obtains the conclusion of Theorem 5.6 under the weaker assumption that the Ricci tensor is parallel for both metrics $g$ and $\hat{g}$. The Nagano-Tanaka approach uses Weyl's normal conformal connection rather than exploiting the local metric structure.

6. The Möbius group. As noted in Theorem 2.2, the Möbius transformations of a Riemannian manifold $(M, g)$ form a group. In the present section we study this group and the Lie algebra of infinitesimal Möbius transformations, giving special attention to the classical case that $M = S^n$. The spaces $\mathcal{U}(M)$ and $\mathcal{U}_g(M)$ and the product $\langle \cdot, \cdot \rangle_k$ play important roles.

The Möbius group contains the group of homotheties and is contained in the group of conformal transformations. Thus,

$$\text{Isom}(M) \subseteq \text{Hty}(M) \subseteq \text{Möb}(M) \subseteq \text{Conf}(M).$$

In a complete, connected manifold other than a Euclidean space, the isometry and homothety groups coincide [Ko]. At the end of the section, we prove a similar result:

THEOREM 6.1. Let $M$ be a complete, connected Riemannian manifold of dimension $n \geq 2$. Then either (a) $\text{Hty}(M) = \text{Möb}(M)$, (b) $\text{Hty}(M)$ has index two in $\text{Möb}(M)$, or (c) $M$ is a standard sphere.

The only instances of (b) are certain warped products $\mathbb{R} \times_P P$ or $\text{Sphere}(f)$ in which $f''/f$ is not constant.

We also consider infinitesimal Möbius transformations, vector fields such that the resulting local flow consists of Möbius transformations. In obvious notation,

$$i(M) \subseteq h(M) \subseteq m(M) \subseteq c(M).$$

Note that these Lie algebras might be larger than the Lie algebras of the corresponding groups, for the vector fields in them, although globally defined, need not be complete.

A vector field $W$ will be infinitesimally conformal if and only if $\nabla X W$ is an infinitesimal conformal transformation of each tangent space; that is,

$$\nabla_X W = \Lambda_W X + \sigma_W(X), \quad X \in TM,$$

(6.1)
where $\Lambda_w = \text{div}(W)/\dim(M)$ and $\sigma_w$ is skew-symmetric. This is equivalent to the usual condition that the Lie derivative of $g$ with respect to $W$ be a functional multiple of $g$.

**Theorem 6.2.** (a) Let $F$ be a conformal transformation of a Riemannian manifold $M$. Then $F$ is isometric, homothetic, or Möbius according as $\|dF\|^{-1}$ is equal to one, a constant, or an element of $\mathcal{U}(M)$.

(b) Let $W$ be an infinitesimal conformal transformation of a Riemannian manifold $M$. Then $W$ is isometric, homothetic, or Möbius according as $\Lambda_w$ is equal to zero, a constant, or an element of $\mathcal{U}(M)$.

**Proof.** For a function $\varphi$ on $M$, we have seen that $B(\varphi) = 0$ if and only if $e^{-\varphi}$ is an element of $\mathcal{U}(M)$. In light of this, part (a) is just a restatement of the definitions.

For part (b) let $\{F_t: t \in \mathbb{R}\}$ be the local flow generated by $W$. Then

$$\|dF_t(x)\| = \exp\left(\int_0^t \Lambda_w(F_s)\, ds\right).$$

From this, it is clear that $W$ is isometric or homothetic according as $\Lambda_w$ is zero or constant. Now suppose that $W$ is Möbius. Since

$$\frac{d}{dt} \bigg|_{t=0} \text{Hess}(\|dF\|^{-1})(X, Y) = \text{Hess} \left( \frac{d}{dt} \bigg|_{t=0} \|dF_t\|^{-1} \right)(X, Y) = -\text{Hess}(\Lambda_w)(X, Y),$$

one sees that $\Lambda_w \in \mathcal{U}(M)$. Conversely, the same computation shows that, if $\Lambda_w \in \mathcal{U}(M)$, then $\mathcal{S}(F_t)$ vanishes to first order in $t$. Since $\mathcal{S}(F_{t+h}) = \mathcal{S}(F_t) + F_* \mathcal{S}(F_h)$, it follows that $\mathcal{S}(F_t)$ vanishes for all $t$, which is to say that $W$ is Möbius. This completes the proof.

We exploit a natural linear representation $\mathcal{S}: \text{Möb}(M) \to \text{Aut}(\mathcal{U}(M))$, given by

$$F_u = (u\|dF\|) \circ F^{-1}, \quad F \in \text{Möb}(M), \quad u \in \mathcal{U}(M).$$

To see that $F_u$ maps $\mathcal{U}(M)$ into itself, note that where $u$ is positive the metric $u^{-2}g$ pushes forward under $F$ to $(F_u)^{-2}g$. Because composites of Möbius transformations are Möbius, this process produces another Möbius metric, and hence $F_u$ solves equation (5.1) on the image. The general result follows since, by adding and subtracting a constant, one can express any element of $\mathcal{U}(M)$ locally as a difference of positive elements.

Similarly, there is a Lie algebra representation of $\mathfrak{m}(M)$ in $\mathcal{U}(M)$, given by

$$W_u = W_u - \Lambda_w u, \quad W \in \mathfrak{m}(M), \quad u \in \mathcal{U}(M).$$

This formula comes from computing the negative of $(d/dt)_{t=0}(F_t)_* u$, where $\{F_t: t \in \mathbb{R}\}$ is the local flow generated by $W$. Since $\mathcal{S}$ is homomorphic, $\mathcal{B}$ is a Lie algebra homomorphism with respect to the usual Lie bracket of vector fields.
We begin with the M"obius group of the standard sphere $S^n$, where $n \geq 2$. In discussions of this group, one often sees an ad hoc realization of $S^n$ as a certain quadric in $\mathbb{R}^{n+1}$. However, there is a completely natural way to the same end, one which explains why the M"obius group of $S^n$ is isomorphic to a group consisting of two of the four components of $O(1, n + 1)$. Recall that $\mathcal{U}(S^n)$ has dimension $n + 2$; in fact, one can specify the values of $u$, $\text{grad } u$, and $\lambda_u$ at any point, and these values determine a unique element of $\mathcal{U}(S^n)$. Also, recall that the product $\langle , \rangle_1$ has signature $(+, - \ldots, -)$. The elements of $\mathcal{U}(S^n)$ are just the restrictions of the affine real-valued functions on $\mathbb{R}^{n+1}$, and the product of such a function $u: x \mapsto p \cdot x + q$ with itself is equal to $q^2 - |p|^2$. However, we do not use this extrinsic description.

For $x \in S^n$ let $\Theta(x)$ be the element of $\mathcal{U}(S^n)$ which vanishes to first order at $x$ with $\lambda_u(x) = 1$.

**Theorem 6.3.** (a) The mapping $\Theta$ embeds $S^n$ anti-isometrically into $\mathcal{U}(S^n)$, with image $\Sigma = \{u \in \mathcal{U}(S^n) : \langle u, u \rangle_1 = 0, \langle u, 1 \rangle_1 = 1\}$.

(b) The representation $\Phi$ takes M"ob($S^n$) isomorphically onto the group $G \subset O(\mathcal{U}(S^n))$ consisting of the orthogonal linear transformations which preserve $\mathbb{R}^+\Sigma$. The preimage of $a \in G$ is the transformation $\Theta^{-1}a'\Theta$, where $a': \Sigma \to \Sigma$ denotes a followed by the radial projection.

**Proof.** From the data for $\Theta(x)$ at $x$, it is clear that $\Theta(x) \in \Sigma$. Let $\gamma$ be a curve in $S^n$ which passes through $x$ with velocity $X$. Differentiating the statements that $\Theta(\gamma(t))$ and its gradient vanish at $\gamma(t)$, one sees that $(d\Theta(x))X$ vanishes at $x$ with gradient equal to $-X$. Therefore

$$\langle (d\Theta)(x)X, (d\Theta)(x)X \rangle_1 = -\|X\|^2.$$  

This shows that $\Theta$ is anti-isometric. Since $\Sigma$ is diffeomorphic to $S^n$, it follows that $\Theta$ is a diffeomorphism onto $\Sigma$.

For part (b), let $F \in \text{M"ob}(S^n)$. To show that $F_\sharp$ is orthogonal, it is enough to show that $\langle u, u \rangle_1 = \langle F_\sharp u, F_\sharp u \rangle_1$ whenever $u$ is positive, for every element of $\mathcal{U}(S^n)$ is locally (and in fact globally) the difference of positive elements. In this case, the equated products are the constant sectional curvatures of $u^{-2}g$ and $(F_\sharp u)^{-2}g$, respectively, and the equality follows from the fact that $F$ is an isometry between these metrics. Using the fact that $\|dF\|^{-1} = F_\sharp^{-1}1$, one sees from the definitions that

$$F_\sharp(\Theta(x)) = \|dF(x)\|^{-1} \Theta(Fx) = \langle 1, F_\sharp(\Theta(x)) \rangle_1 \Theta(Fx).$$

Thus $F_\sharp$ preserves $\mathbb{R}^+\Sigma$, so that $F_\sharp \in G$, and $\Theta^{-1}F_\sharp \Theta = F$. The latter equation shows that $\sharp$ is injective.

Consider any element $a \in G$. Since the radial direction is orthogonal to all tangent vectors to the light cone, one sees that $a'$ is conformal, with

$$\|a'(\Theta(x))\|^{-1} = \langle a(\Theta(x)), 1 \rangle_1 = \langle \Theta(x), a^{-1}1 \rangle_1 = (a^{-1}1)(x), \quad x \in S^n.$$  

The transformation $F = \Theta^{-1}a'\Theta$ in $S^n$ is therefore M"obius, with $\|dF\|^{-1} = a^{-1}1$. 
Finally, \( F_z \) equals \( a \), for by equation (6.4) these transformations agree on the spanning set \( \Sigma \). This completes the proof.

Every element of \( \text{M"{o}b}(S^n) \) can be expressed uniquely as the product of an isometry and a \textit{longitudinal} transformation. The latter can be defined in several equivalent ways. For example, a transformation is longitudinal if and only if it is isometrically conjugate to a dilation in \( \mathbb{R}^n \subseteq S^n \), i.e., to a mapping which in stereographic coordinates has the form \( x \mapsto cx \), \( c > 0 \). We simply construct certain mappings \( F_v \), parametrized by the positive sheet \( \langle v, 1 \rangle_1 > 0 \) of the hyperboloid \( \langle v, v \rangle_1 = 1 \), and declare these to be the longitudinal transformations. The main feature of the construction is that \( \|dF_v\|^{-1} = v \). Since every function \( \|dF\|^{-1} \), \( F \in \text{M"{o}b}(S^n) \), lies in the positive sheet of the hyperboloid, it is clear that unique factorization obtains as claimed. The reader might recognize our polar decomposition as arising from the Cartan involution \( F \mapsto RFR \) of \( \text{M"{o}b}(S^n) \), where \( R \) is the antipodal mapping in \( S^n \). This is compatible with the Iwasawa decomposition \( \text{M"{o}b}(S^n) = KAN \), where \( K = O(n+1) \), \( A \) is the one-parameter group of dilations in \( \mathbb{R}^n \), and \( N \) is the group of translations in \( \mathbb{R}^n \).

For \( v \) as above, we construct an element \( a_v \in G \) such that \( a_v^{-1}1 = v \) and define \( F_v = \Theta^{-1}a_v\Theta \). The simplest candidate is the reflection in the line through \( v + 1 \). However, this would result in the preferred isometry \( F \) being the antipodal mapping \( R \). We therefore define \( a_v \) to be this reflection followed by the reflection in the line through 1. Thus

\[
a_v: u \mapsto u - \frac{2\langle u, v + 1 \rangle_1}{\langle v + 1, v + 1 \rangle_1} (v + 1) + 2 \langle u, v \rangle_1 1, \quad u \in \mathcal{U}(S^n).
\]

For a point \( b \) in the open unit ball in \( \mathbb{R}^{n+1} \), let

\[
v_b(z) = \frac{1 + |b|^2}{1 - |b|^2} + \frac{2}{1 - |b|^2} (b \cdot z), \quad z \in S^n.
\]

This construction parametrizes the family \( \{v\} \), and by direct computation one finds that

\[
F_v: z \mapsto \frac{(1 - |b|^2)z + 2(1 + (b \cdot z))b}{1 + 2(b \cdot z) + |b|^2}.
\]

Thus longitudinal transformations generalize the transformations \( z \mapsto (z + b)/(1 + bz) \) of the unit circle \( S^1 \subseteq C \).

The ideas behind Theorem 6.3 apply to any simply connected space \( M \) of constant curvature. Recall that, by Theorem 5.1, the space \( \mathcal{U}(M) \) has the same key features as does \( \mathcal{U}(S^n) \); values of \( u \), grad \( u \), and \( \lambda_u \) can be specified at any point and uniquely determine an element of \( \mathcal{U}(M) \), and the product \( \langle \ , \rangle_k \) has signature
THE SCHWARZIAN DERIVATIVE

(+, −, . . . , −). Defining Θ as before, one sees that it is an anti-isometric immersion into \( \mathcal{U}(M) \) whose image is contained in the intersection \( \Sigma \) of the light cone \( \langle u, u \rangle_k = 0 \) with the affine hyperplane \( \langle u, 1 \rangle_k = 1 \). However, \( \Sigma \) is not always diffeomorphic to a sphere; in fact, it is a paraboloid if \( K = 0 \) and is a hyperboloid of two sheets if \( K < 0 \). (One can picture this by fixing a Lorentzian inner product space and imagining a vector \( v \) which moves from the interior to the exterior of the light cone as a parameter \( K \) varies from positive to negative; the hyperplane \( \langle u, 1 \rangle = 1 \) rotates in the opposite direction, cutting the light cone more and more obliquely until, when \( K < 0 \), it cuts both halves.) The same arguments as before show that \( \Phi \) maps into the group \( G \) of orthogonal transformations in \( \mathcal{U}(M) \) which preserve \( \mathbf{R}^+ \Theta(M) \) and that \( F \Phi = \Theta F \) when \( F \in \text{Möb}(M) \).

To say more, one must assume that \( \Theta \) is injective. The light cone quadric \( S \) in the projective space of \( \mathcal{U}(M) \) then serves as a natural “Möbius completion” of \( M \); it is a Möbius \( n \)-sphere, the Möbius-equivalence class of metrics coming from the hyperplane slices of the light cone, and the projection from \( \Theta(M) \) into \( S \) is a Möbius embedding. The same arguments as before show that \( \Phi \) is an isomorphism onto \( G \) in this case. Hence one can identify \( \text{Möb}(M) \) with the group of Möbius transformations of \( S \) which leave the image of \( M \) invariant.

When \( M \) is complete as well as simply connected, \( \Theta \) maps diffeomorphically onto a component of \( \Sigma \). If \( K = 0 \), the construction then gives a natural way of adding the point at infinity, i.e., the line through 1 in \( \mathcal{U}(M) \), to form a Möbius sphere. Since \( G \) must leave this line invariant, one sees incidentally that every Möbius transformation of \( M \) is homothetic. Similarly, if \( K < 0 \), the construction reveals that:

1. Every Möbius transformation of \( M \) is isometric.
2. \( M \) has an ideal boundary \( S' \), a Möbius \( (n - 1) \)-sphere, such that \( \text{Möb}(M) \) is naturally isomorphic to \( \text{Möb}(S') \).

For these, let \( S' \) be the boundary of the image of \( M \) in \( S \). This is the light cone quadric for the Lorentzian space \( \mathcal{P} = 1^+ \subseteq \mathcal{U}(M) \) and so is a Möbius \( (n - 1) \)-sphere. We have seen that the Möbius group of \( S \) corresponds to the group \( G' \) of orthogonal linear transformations of \( \mathcal{P} \) which preserve time orientation. Now, \( G \) consists of the orthogonal transformations of \( \mathcal{U}(M) \) which preserve the set \( \langle u, 1 \rangle_k > 0 \) in one half of the light cone. These are precisely the transformations which preserve \( \mathcal{P} \), preserve time-orientation therein, and preserve orientation in the quotient \( \mathcal{U}(M)/\mathcal{P} \). Such a mapping must be the identity on the line through 1. This yields assertion (1); furthermore, it shows that restriction to \( \mathcal{P} \) defines an isomorphism of \( G \) with \( G' \). One sees, then, that every Möbius transformation of the image of \( M \) in \( S \) extends to the boundary \( S' \) and that this correspondence defines an isomorphism of \( \text{Möb}(M) \) with \( \text{Möb}(S') \).

The proof of Theorem 6.3 also yields a theorem of Liouville: A locally defined conformal transformation in \( \mathbf{R}^n \), \( n \geq 3 \), is either a homothety or the composite of a homothety with an inversion through some sphere. This conclusion certainly holds for global Möbius transformations of \( S^* \) since, as noted above, any Möbius transformation in \( \mathbf{R}^* \) is homothetic, and an inversion can map any point to \( \infty \). Thus Liouville’s theorem is a consequence of the following result.
THEOREM 6.4. Let $U$ be a connected open set in $S^n$, where $n \geq 2$, and let $F: U \to S^n$ be a M"obius local diffeomorphism. Then $F$ extends to a global M"obius automorphism of $S^n$. In particular, this holds when $F$ is conformal and $n \geq 3$.

For this, the key observation is that $F_\sharp$ makes sense as an operator in $\mathcal{U}(S^n)$. Any solution of the M"obius equation on a connected open set in $S^n$ extends uniquely to an element of $\mathcal{U}(S^n)$. One can therefore define $F_\sharp u$ by restricting $u$ to a connected open set $V$ on which $F$ is injective, applying $F_\sharp$ to the restriction and extending; since $U$ is connected, the result is independent of $V$. The same argument as above shows that $F_\sharp \in G$ and that $F_\sharp \Theta = \Theta F$ on $U$. Therefore, $\Theta^{-1}F_\sharp \Theta \in \text{M"ob}(S^n)$ extends $F$.

In the classical setting of complex analysis, one considers orientation-preserving conformal mappings defined on connected open sets in the Riemann sphere $S^2$. Here, the M"obius transformations $f$ can be characterized in four different ways:

(a) $f(z) = (az + b)/(cz + d)$ for some complex numbers $a, b, c, d$.
(b) $f$ takes circles, including lines, to circles.
(c) The Schwarzian derivative $S(f)$ vanishes.
(d) $f$ extends to a conformal automorphism of $S^2$.

The first of these is not geometric. However, we have just seen how M"obius transformations, in the sense of vanishing Schwarzian tensor, correspond to a group consisting of two of the four components of $O(\mathcal{U}(S^2)) \cong O(1, 3)$ and, furthermore, how to produce explicit formulas in terms of $O(3)$ and the unit ball in $\mathbb{R}^3$. Admittedly, this falls short of explaining why the identity component should be isomorphic to $\text{PSL}(2, \mathbb{C})$. The equivalence of (b) and (c), and more generally the relation between the Schwarzian tensor and the second fundamental form, is the theme of Section 3. Theorem 6.4 asserts that (c) implies (d). The reverse implication is perhaps less obvious. However, one can show that, if $p$ is the Schwarzian tensor of a conformal automorphism of $S^2$, then $\Delta(\log \|p\|) = 2$ wherever $p$ is nonzero. Since $\|p\|$ attains a maximum, it must vanish, and hence $f$ is necessarily M"obius.

We now turn to infinitesimal M"obius transformations. One could base this study on local analysis in terms of warped product structures, along the lines of Section 5. The reader interested in this approach will have little trouble formulating the infinitesimal analogue of Lemma 6.6 below. However, we work globally, attempting to see what one can say about $m(M)$ from knowledge of $\mathcal{U}(M)$. Recall that for a connected space of dimension $n \geq 2$ exactly one of the following holds:

A. $\mathcal{U}(M) = \mathbb{R}$.
B. $\mathcal{U}(M)$ is two-dimensional, and $\mathcal{U}_k(M) = \mathbb{R}$ for all $K$.
C. $\mathcal{U}(M)$ has dimension at least two, and $\mathcal{U}(M) = \mathcal{U}_k(M)$ for some $K$.

The real information here, that if $\mathcal{U}(M)$ is at least three-dimensional then $\mathcal{U}(M) = \mathcal{U}_k(M)$ for some $K$, follows from part (b) of Theorem 5.2.

Let $\mathfrak{g}$ be the kernel of the Lie algebra representation $\mathfrak{g}$. From the identity $Z_1 = -\Lambda_2$, one sees that $\mathfrak{g}$ consists of the infinitesimal isometries $Z$ such that $Z u = 0$ for all $u \in \mathcal{U}(M)$. Our goal is to describe the structure of $m(M)$ up to the internal structure of $\mathfrak{g}$.

THEOREM 6.5. Let $M$ be a connected Riemannian manifold of dimension $n \geq 2$. 

THE SCHWARZIAN DERIVATIVE

(a) If $M$ is of type $A$, then $3 = i(M)$ and $b(M) = m(M)$.

(b) If $M$ is of type $B$, then $3 = i(M)$, and $m(M)/3$ is at most one-dimensional.

(c) Suppose that $M$ is of type $C$. Then $\mathfrak{h}$ maps $m(M)$ into the Lie algebra $\mathfrak{o}(\mathfrak{U}(M))$ of anti-self-adjoint linear endomorphisms of $\mathfrak{U}(M)$. The vector fields $b \operatorname{grad} a - a \operatorname{grad} b, a, b \in \mathfrak{U}(M)$, span an ideal $a$ in $m(M)$, and $\mathfrak{h}$ takes an isomorphically onto the Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{o}(\mathfrak{U}(M))$ consisting of the endomorphisms which annihilate $\mathfrak{U}(M)^{\perp}$. In particular, if the product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ is nondegenerate, then $a \cong \mathfrak{o}(\mathfrak{U}(M))$ and $m(M)$ is the Lie algebra direct sum $a \oplus 3$.

Before launching the proof, we note what is not said here. In part (a) we do not say whether $b(M)$ coincides with $i(M)$; sometimes it does, and sometimes it does not. Of course, the dimension of $b(M)/i(M)$ never exceeds one.

Part (b) asserts only that the total degree of the extension $i(M) \subseteq b(M) \subseteq m(M)$ is at most one. This allows three possible combinations, each of which can indeed occur. For example, consider a warped product $I \times_f \mathbb{R}$, where $I$ is an interval. It can be shown that the infinitesimal Möbius transformations in such a space are the vector fields

$$W(t, y) = h(t) \frac{\partial}{\partial t} + (ay + b) \frac{\partial}{\partial y}, \quad a, b \in \mathbb{R}, \quad h' = a + h''/f, \quad h''/f = \text{constant}.$$ 

Using this, one can verify the following as examples:

- $i = \mathfrak{h} = m$: $f(t) = te^t, t > 0$;
- $i = \mathfrak{h} \neq m$: $f = r''$, $t \approx 0$, where $r''' = r''(r' + 1)/r, r(0) = 1, r'(0) = 0, r''(0) = 1$;
- $i \neq \mathfrak{h} = m$: $f(t) = t^2, t > 0$.

Part (c) is satisfactory when the product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ is nondegenerate. This occurs, for example, when $M$ is a simply connected space of constant curvature; here, $\mathfrak{s}$ vanishes since the gradients of the elements of $\mathfrak{U}(M)$ span each tangent space, and hence $m(M)$ is isomorphic to $\mathfrak{o}(\mathfrak{U}(M))$. For a general manifold of type $C$, the product can be degenerate. However, $\mathfrak{U}(M)^{\perp}$ is at most one-dimensional, as one sees by evaluating products at a point. Since $\mathfrak{U}(M)^{\perp}$ is invariant under $\mathfrak{o}(\mathfrak{U}(M))$, the unresolved issue in the degenerate case is whether the representation of $m(M)$ on the line $\mathfrak{U}(M)^{\perp}$ is trivial. If it is, then $m(M) = a \oplus \mathfrak{s}$; if it is not, then $m(M)/(a \oplus \mathfrak{s})$ is one-dimensional. As in part (b), both possibilities occur. Examples are provided by products $\mathbb{R} \times P$, where $\mathfrak{U}_0(P) = \mathbb{R}$. Here, one can show that $m(M)$ acts trivially on $\mathfrak{U}(M)^{\perp}$ if and only if $b(P) = i(P)$.

**Proof of Theorem 6.5.** Part (a) follows from Theorem 6.2(b) and the definition of $\mathfrak{s}$.

For part (b), fix a nonconstant function $u \in \mathfrak{U}(M)$. If $W \in m(M)$ with $\Lambda_W = au + b$ and $W_u = cu + d$, then by differentiating $W_u$ with respect to $\operatorname{grad} u$ one finds that

$$\lambda_u(au^2 + (b + c)u + d) - au\|\operatorname{grad} u\|^2 = c\|\operatorname{grad} u\|^2.$$
Suppose that $W \in i(M)$, so that $a$ and $b$ vanish. If $c$ and $d$ did not both vanish, equation (6.5) would imply that $\lambda_\alpha$ was a constant multiple of $cu + d$, contrary to the assumption that $M$ is of type B. Thus $W_s u = 0$, and so $W \in \mathcal{S}$. Since $\mathcal{S} \subseteq i(M)$ in general, this establishes the first assertion of part (b). In light of this, the second assertion is that the range of the mapping $W \mapsto \Lambda_W$ is at most one-dimensional. Suppose this were not the case. One could then find vector fields $W, W' \in m(M)$ with $\Lambda_W = u$ and $\Lambda_{W'} = 1$. Eliminating the term $\|\text{grad } u\|^2$ from the corresponding equations (6.5) would yield

$$\lambda_\alpha(u^2 + (c + d')u + (cd' - c'd)) = 0.$$  

But this is impossible; $\lambda_\alpha$ is not identically zero since $M$ is of type B, and $u$ is not constant on any open set. The contradiction completes the proof of part (b).

For part (c), we again differentiate $W_s u$ with respect to $\text{grad } u$, obtaining

$$u^2 \langle \Lambda_W, 1 \rangle_K + u(\langle W_s u, 1 \rangle_K - \langle \Lambda_W, u \rangle_K) - \langle W_s u, u \rangle_K = 0, \quad u \in \mathcal{U}(M),$$

when $W \in m(M)$. If $u$ is not constant, then all coefficients must vanish, and by continuity they vanish identically in $\mathcal{U}(M)$. From the last coefficient, one sees that $W_s \in o(\mathcal{U}(M))$. Let $\rho: \mathcal{U}(M) \wedge \mathcal{U}(M) \to \text{Vect}(M)$ be the linear mapping which takes $a \wedge b$ to the vector field $b \text{grad } a - a \text{grad } b$; by definition, $a$ is the image of $\rho$. Since

$$\nabla_x (\rho(a \wedge b)) = (b \lambda_a - a \lambda_b) X + ((Xb) \text{grad } a - (Xa) \text{grad } b), \quad X \in TM,$$

and since $b \lambda_a - a \lambda_b = b \langle a, 1 \rangle_K - a \langle b, 1 \rangle_K$, one sees that $\rho(a \wedge b)$ is an infinitesimal Möbius transformation. The following identities are easily checked:

$$[W, \rho(a \wedge b)] = \rho(W_s a \wedge b) + \rho(a \wedge W_s b), \quad W \in m(M),$$

$$\rho(a \wedge b)_s u = a \langle b, u \rangle_K - b \langle a, u \rangle_K, \quad u \in \mathcal{U}(M).$$

From the former, one sees that $a$ is an ideal in $m(M)$. Now, it is an exercise in linear algebra to verify that the endomorphisms $u \mapsto a \langle b, u \rangle_K - b \langle a, u \rangle_K$ span the Lie algebra $\mathfrak{g}$ of anti-self-adjoint endomorphisms which annihilate $\mathcal{U}(M)^+$, regardless of the signature of the product. Since $\mathcal{U}(M)^+$ is at most one-dimensional, a count of dimensions then shows that $\mathfrak{g}$ is an isomorphism from $\mathcal{U}(M) \wedge \mathcal{U}(M)$ onto $\mathfrak{g}$, and hence that $\mathfrak{g}$ is an isomorphism from $a$ onto $\mathfrak{g}$. This completes the proof of Theorem 6.5.

One can use equation (5.3) and part (c) of this theorem to obtain explicit formulas for the infinitesimal Möbius transformations in $\mathbb{R}^n$ or, equivalently, in $S^n$; they are the vector fields

$$W(x) = v + (c + Q)x + (w \cdot x)x - \frac{1}{2} |x|^2 w, \quad x \in \mathbb{R}^n,$$
with \( v, w \in \mathbb{R}^n, c \in \mathbb{R}, \) and \( Q \in o(n) \). If the function \( u \) in equation (5.3) is identified with the column vector \((2A, B, C)\) of its coefficients, then the representation \( \rho \) takes the matrix form

\[
W \mapsto \begin{pmatrix}
  c & -w^T & 0 \\
v & -Q & -w \\
0 & v^T & -c
\end{pmatrix}.
\]

Recall, however, that the derivative of the group representation \( \rho \) takes \( W \) to the negative of this matrix.

We end this section by proving Theorem 6.1. Let \( M \) be a complete, connected space of dimension \( n \geq 2 \). We assume that \( \mathcal{H}(M) \neq \mathbb{R} \), since otherwise it is clear that \( \mathcal{H}(M) \) equals \( \text{M\" ob}(M) \). By Theorem 5.4, \( M \) is isometric to one of the following:

I. a complete, simply connected space of constant curvature;

II. with \( P \) complete and connected and \( 1 \leq m < n \)
   A. \( \mathbb{R}^n \times P \), where \( \mathcal{U}_0(P) = \mathbb{R} \),
   B. \( \mathcal{H}(K) \times f P \), where \( f(x) = 1/x_m \) and \( \mathcal{U}_0(P) = \mathbb{R} \), or
   C. \( \mathcal{H}(K) \times f P \), where \( f(x) = (|x|^2 + 1)/(2x_m) \) and \( \mathcal{U}_K(P) = \mathbb{R} \);

III. with \( f^\prime/f \) not constant
   A. \( \mathbb{R} \times f P \), where \( P \) is complete and connected,
   B. \( \text{Polar}(f) \), where \( T = \infty \), or
   C. \( \text{Sphere}(f) \).

We have seen that in a simply connected space of constant curvature \( K \leq 0 \) the homothety and Möbius groups coincide. One can also see this by the following direct argument, which applies to spaces of type II as well. Let \( F \) be a Möbius transformation of \( M \), where \( \mathcal{H}(M) = \mathcal{H}_K(M) \) with \( K \leq 0 \). Along a unit-speed geodesic \( \gamma \) in \( M \), the function \( u = \|dF\|^{-1} \) satisfies the equation \( u'' + Ku' = 0 \). By examining the positive solutions of this equation, one sees that if \( u(t) \) were not constant, then at least one end of \( \gamma \) would have finite length in the metric \( u^2 = F^*g \), contradicting the completeness of this metric. It follows that \( u \) is constant along all geodesics, and hence that \( F \) is homothetic.

We assume, therefore, that \( M \) is of type III. If it is of type IIIIB, then the pole is the only point at which the gradients of all elements of \( \mathcal{H}(M) \) vanish. It easily follows that every Möbius transformation of \( M \) fixes the pole. Similarly, if \( M \) is of type IIIIC, then every Möbius transformation either fixes or permutes the two poles. This being understood, we treat the three subtypes simultaneously, considering the space \( M' \) obtained by deleting any poles. Note that \( M' \) is a warped product \( I \times f P \), where \( f''/f \) is not constant. The interval \( I \) is equal to \( \mathbb{R}, (0, \infty) \), or a finite interval \( (0, T) \) according as \( M \) is of type IIIA, IIIIB, or IIIIC, and in the latter two cases \( P = S^{n-1} \).

**Lemma 6.6.** Let \( M' = Q \times f P \), where \( Q \) and \( P \) are connected Riemannian manifolds of dimension at least one. Suppose that every element of \( \mathcal{H}(M') \) is constant on fibers \( \{q\} \times P \) and that \( \mathcal{H}(M') \) separates fibers. Then the Möbius transformations of \( M' \) are the mappings \( F(q, p) = (G(q), H(p)) \) such that...
(1) \( G \in \text{Conf}(Q) \) and \( H \in \text{Hty}(P) \),
(2) \( f(q)\|dG(q)\| = f(G(q))\|dH\| \) for all \( q \in Q \), and
(3) the function \( (q, p) \mapsto \|dG(q)\|^{-1} \) is an element of \( \mathcal{U}(M') \).

Proof. Let \( F \) be a Möbius transformation of \( M' \). The identity \( F_{\ast}^{-1}u = \|dF\|^{-1} \)
\( (u \circ F) \) shows that \( u \circ F \) is constant on fibers for all \( u \in \mathcal{U}(M') \). Since \( \mathcal{U}(M') \) separates fibers, it follows that \( F \) takes fibers into fibers. This, in turn, implies that \( F \) also takes the horizontal slices \( Q \times \{p\} \), which are orthogonal to the fibers, into horizontal slices. Thus one has only to consider transformations of the form \( F(q, p) = (G(q), H(p)) \). For such a transformation to be conformal, it is necessary and sufficient that \( G \) and \( H \) be conformal with

\[
f(q)\|dG(q)\| = f(G(q))\|dH(p)\| \quad \text{for all } (q, p) \in M'.
\]

Clearly, then, conformality obtains if and only if conditions (1) and (2) hold. Since \( \|dF(q, p)\| = \|dG(q)\| \), conditions (3) is the Möbius condition.

In the present argument, Lemma 6.6 says that the Möbius transformations of \( M' \) are the mappings \( F(t, p) = (G(t), H(p)) \) with:

1. \( H \in \text{Hty}(P) \); let \( c = \|dH\| \).
2. \( f(t)|G'(t)| = cf(G(t)) \) for all \( t \in I \).
3. \( \langle 1/G' \rangle = af \) for some \( a \in \mathbb{R} \).

Lemma 6.7. If \( G \) preserves orientation, then it is an isometry.

Proof. Suppose that \( I = \mathbb{R} \) and that \( a \) is not zero. Since

\[
\frac{d}{dt} \left( \frac{c}{G'(G(t))} + G'(t) \right) = aG'(t)(cf(G(t)) - f(t)G'(t)) = 0,
\]

the expression being differentiated has a constant value \( b \). The following diagram then commutes:

\[
\begin{array}{ccc}
I & \overset{G}{\longrightarrow} & I \\
\downarrow{G'} & & \downarrow{G'} \\
\mathbb{R} \cup \{\infty\} & \overset{r}{\longrightarrow} & S^1,
\end{array}
\]

Since \( b \) and \( c \) are positive, \( \Gamma \) is an orientation-preserving diffeomorphism of \( S^1 \) with zero, one, or two fixed points; if it has two, they are positive numbers \( x_- < x_+ \), with \( x_- \) an attractor and \( x_+ \) a repellor, whose product is \( c \). Now, \( G' \) is an open mapping. Since \( G'(I) \) is \( \Gamma \)-invariant and does not pass through \( \infty \), it must be that \( \Gamma \) has two fixed points, and \( G'(I) \) is the interval between them. Note that \( G \) can have no fixed point, since \( \Gamma \) has none in \( G'(I) \). Let \( d_n = |G^{n+1}(0) - G^n(0)| \). The quotient \( d_{n+1}/d_n \),
being the average value of \( G' \) on the segment between \( G_{n+1}(0) \) and \( G'(0) \), approaches \( x_- \) as \( n \) tends to infinity and approaches \( x_+ \) as \( n \to -\infty \). If \( c \leq 1 \), then \( x_- < 1 \), and hence the sequence \( G'(0) \) converges geometrically as \( n \to \infty \). Similarly, if \( c > 1 \), then \( x_+ < 1 \) and this sequence converges as \( n \to -\infty \). In either case the limit must be a fixed point of \( G \), but none exists.

The contradiction shows that if \( I = \mathbb{R} \), then \( a = 0 \); that is, \( G' \) is constant. Suppose that this constant value is something other than one. Then \( G \) has a fixed point \( t_0 \) which is either a global attractor or a global repellor. Setting \( t = t_0 \) in equation (2), one sees that the constant value of \( G' \) is equal to \( c \); letting \( t \) vary, one then sees that \( f(G(t)) = f(t) \) for all \( t \). Since \( t_0 \) is a point of closure of all the orbits of \( G \), it follows that \( f \) is constant, contradicting the fact that \( f''/f' \) is not constant. Therefore, \( G' \) is identically equal to one, as was to be proved.

Now suppose that \( I = (0, \infty) \) or \( I = (0, 1) \). Here, \( c = 1 \), since every homothety of \( S^{n-1} \) is an isometry. Differentiating equation (2), one finds that

\[
 f'(G(t)) - f'(t) = -af(t)f(G(t)).
\]

Differentiating this equation and using (2) again, one obtains

\[
 f''(G(t)f(G(t)) - f'(G(t))^2 = f''(t)f(t) - f'(t)^2.
\]

If \( G \) had no fixed point, the closure in \( \mathbb{R} \) of each orbit of \( G \) would contain zero. Since \( f''f - (f')^2 \) is continuous on \( I \cup \{0\} \), this would imply that \( f''f - (f')^2 \) was constant, contradicting the fact that \( f''/f' \) is not constant. Therefore \( G \) has a fixed point \( t_0 \). With \( t = t_0 \), equation (2) shows that the constant value of \( G' \) is equal to \( 1 \). This completes the proof.

Consider the group of base transformations \( G \) arising from the elements of \( \text{M"ob}(M) \). By Lemma 6.7, it is clear that those which are isometric constitute a subgroup of index at most two. Since the mapping \( F \mapsto G \) is homomorphic, and since \( F \) is isometric if and only if \( G \) is, it follows that \( \text{Isom}(M) \) has index at most two in \( \text{M"ob}(M) \). This completes the proof of Theorem 6.5 and shows incidentally that \( \text{Isom}(M) \) equals \( \text{Hty}(M) \) when \( M \) is of type III.

The homothety group has index two if and only if there exists \( F \in \text{M"ob}(M) \) such that the base transformation \( G \) reverses orientation and is nonsymmetric. This cannot occur in type IIIb, where every base transformation preserves orientation. To construct examples in the other types, it is helpful first to examine what \( F \) would be like if it did exist. Let \( t_0 \) be the fixed point of \( G \). Since \( G^2 \) fixes \( t_0 \), preserves orientation, and is covered by \( F^2 \), it must be the identity. Setting \( t = t_0 \) in equation (2), one finds that \( c = 1 \). Now consider the metric \( g_1 = u^{-2}g \) in \( M \), where \( u(t, p) = 1 - 1/G(t) \). This is a M"obius change of metric, and one sees that \( F \) is an isometry of \( g_1 \). Provided such a transformation \( F \) exists, then, its existence can be explained by the fact that some M"obius change of metric endows \( M \) with additional symmetry.
One can construct examples by reversing this thought. For example, let \((M, g_1) = \text{Sphere}(f_1)\), where \(f_1\) is symmetric about \(T/2\). The metric

\[
g = v^{-2} g_1, \quad v(t, p) = 1 + \int_0^t f_1(s) \, ds,
\]

is then a spherical warped product, and the reflection about \(T/2\) is a nonhömtethic Möbius transformation in \(g\). For an example of type IIIA, one could choose \(f_1\) to be symmetric about zero with \(\int_0^T f_1(s) \, ds < 1\).

References


THE SCHWARZIAN DERIVATIVE


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