HYPERBOLIC CURVATURE AND CONFORMAL MAPPING

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1. *Introduction*

The connection between the second derivative of a conformal mapping and curvature has been used in a number of ways. In this note we give an intrinsic formulation of this as a kind of Schwarz lemma for hyperbolic curvature. Throughout, D will denote a simply connected domain in $\mathbb{C}$ with at least two boundary points and $\kappa_D(z,\gamma) = \kappa_{\lambda_D}(z,\gamma)$ will denote the geodesic curvature of a smooth curve γ in D at $z \in \gamma$ measured in the hyperbolic metric $\lambda_D|dz|$ of D (constant curvature -1).

THEOREM 1. *If f is a conformal mapping of D into itself then*

$$\max\{\kappa_D(f(z), f(\gamma)), 2\} \geqslant \max\{\kappa_D(z,\gamma), 2\}.$$

It will be seen that this is equivalent to the classical coefficient inequality $|a_2| \leqslant 2$ for the class S and that the constant 2 is the 'same', thus establishing the sharpness. What we shall show is that $|a_2| \leqslant 2$ is equivalent to the Schiffer–Tammi inequality for functions mapping the unit disc into itself, which in turn is equivalent to Theorem 1. In §3 we characterize curves γ with $\kappa_D(z,\gamma) \geqslant 2$ for *all* $z \in \gamma$ in terms of their mapping properties and their euclidean geometry.

We recall a few necessary facts about geodesic curvature. As a generalization of euclidean curvature, geodesic curvature can be most directly defined as $d\theta/ds$ where s is arclength in the metric and θ is the angle between the tangent vector field to the curve and any vector field along the curve which remains parallel. If $\rho|dz|$ is any smooth conformal metric on D then the euclidean and ρ-geodesic curvatures of a smooth curve $\gamma \subset D$ satisfy

$$\kappa_e(z,\gamma) = \rho(z)\,\kappa_\rho(z,\gamma) + \frac{\partial}{\partial n}\log\rho(z). \tag{1}$$

Here $\partial/\partial n$ is the normal derivative $+\pi/2$ from the tangent vector; see [**1**, p. 329]. Note that in (1) we regard κ_e and κ_ρ as functions of the points z on the curve rather than as functions of euclidean and ρ-arclength parameters respectively. As the statement of Theorem 1 would indicate, this is a useful convention. The geodesic curvature is a conformal invariant, so that $\kappa_{\hat\rho}(z,\gamma) = \kappa_\rho(f(z), f(\gamma))$ whenever $\hat\rho = (\rho\circ f)|f'|$ is the pullback of ρ under a (locally) conformal mapping f. This invariance and (1) yield the familiar formula for the change in euclidean curvature,

$$\kappa_e(z,\gamma) = \kappa_e(f(z), f(\gamma))\,|f'(z)| + \frac{\partial}{\partial n}\log|f'|(z). \tag{2}$$

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Here $\hat{\rho} = |f'|$ is the pullback of the euclidean metric $\rho = 1$. Of course (2) can be derived easily from scratch with no mention of geodesic curvature, but the approach through conformal invariance has its advantages.

Finally, we recall the relevant coefficient inequalities. As already mentioned, if $g(z) = z + a_2 z^2 + \ldots$ is a normalized conformal mapping of the unit disc B, then $|a_2| \leqslant 2$ and equality holds only for rotations of the Koebe function, $k_\theta(z) = z(1-e^{i\theta}z)^{-2}$. If $f(z) = b_1 z + b_2 z^2 + \ldots$ is a conformal mapping of B into itself then the Schiffer–Tammi inequality states that

$$|b_2| \leqslant 2|b_1|(1-|b_1|). \tag{3}$$

This is usually proved using the Grunsky inequalities as in [3, Chapter 4] but a simple observation, which may go back to Pick, shows it to be a consequence of $|a_2| \leqslant 2$. For if f is as above, consider

$$h(z) = e^{-i\phi} b_1^{-1} k_\theta(f(e^{i\phi}z)) = z + [2e^{i(\theta+\phi)} b_1 + e^{i\phi} b_2/b_1] z^2 + \ldots$$

where ϕ and θ are chosen so that both terms in the bracket are positive. Then (3) follows from applying $|a_2| \leqslant 2$ to h. Conversely, by approximating a general conformal mapping $g(z) = z + a_2 z^2 + \ldots$ of B by bounded conformal mappings one sees that (3) implies $|a_2| \leqslant 2$.

2. *Proof of Theorem* 1

The conformal invariance of hyperbolic curvature allows us to consider only the case when D is the unit disc B, $z = 0$ and $f(z) = b_1 z + b_2 z^2 + \ldots$. The hyperbolic metric for B has $\lambda_B(z) = 2(1-|z|^2)^{-1}$ and $\nabla \log \lambda_B(0) = 0$, hence (1) gives for any smooth curve Γ through 0, $2\kappa_B(0,\Gamma) = \kappa_e(0,\Gamma)$. Thus we must show that $\kappa_e(0, f(\gamma)) \geqslant \kappa_e(0,\gamma)$ when $\kappa_e(0,\gamma) \geqslant 4$. But (2) and (3) imply that

$$\begin{aligned}\kappa_e(0, f(\gamma)) &= |f'(0)|^{-1}\left(\kappa_e(0,\gamma) - \frac{\partial}{\partial n}\log|f'|(0)\right)\\ &\geqslant |b_1|^{-1}(\kappa_e(0,\gamma) - 2|b_2|/|b_1|)\\ &\geqslant |b_1|^{-1}(\kappa_e(0,\gamma) - 4(1-|b_1|)\end{aligned}$$

and the last line is at least $\kappa_e(0,\gamma)$ since $\kappa_e(0,\gamma) \geqslant 4$.

We complete the chain of implications back to $|a_2| \leqslant 2$ by showing that Theorem 1 implies the Schiffer–Tammi inequality (3). For this, let $f(z) = b_1 z + b_2 z^2 + \ldots$ be a conformal mapping of B into itself. Let $\gamma \subset B$ be a curve through the origin with $\frac{1}{2}\kappa_e(0,\gamma) = \kappa_B(0,\gamma) = 2$. Then by Theorem 1

$$\tfrac{1}{2}\kappa_e(0, f(\gamma)) = \kappa_B(0, f(\gamma)) \geqslant \kappa_B(0,\gamma) = \tfrac{1}{2}\kappa_e(0,\gamma).$$

Using equation (1) as in the above proof we find that

$$\frac{\partial}{\partial n}\log|f'|(0) = \kappa_e(0,\gamma) - \kappa_e(0, f(\gamma))|f'(0)| \leqslant 4(1-|f'(0)|).$$

As this holds for any normal direction the inequality $|b_2| \leqslant 2|b_1|(1-|b_1|)$ follows. The claim that 2 is sharp in Theorem 1 is now settled.

We next give an equivalent statement of Theorem 1 which shows the monotonicity of the hyerbolic curvature of a fixed curve when the containing domain decreases.

COROLLARY 1. *If G is a simply connected subdomain of D and if $\gamma \subset G$ then*

$$\max\{\kappa_D(z,\gamma), 2\} \geqslant \max\{\kappa_G(z,\gamma), 2\}.$$

The constant 2 *is sharp.*

We observe that this is in contrast to the inequality between the total hyperbolic curvatures of a Jordan curve in the two domains. Indeed, the Gauss–Bonnet theorem together with $\lambda_G \geqslant \lambda_D$ imply that

$$\int_\gamma \kappa_D(z,\gamma)\,\lambda_D(z)\,|dz| \leqslant \int_\gamma \kappa_G(z,\gamma)\,\lambda_G(z)\,|dz|,$$

whenever γ is a Jordan curve in G.

The statement of Corollary 1 also allows us to deduce a simple sufficient condition, in euclidean terms, for a curve to have hyperbolic curvature at least 2.

COROLLARY 2. *If $\kappa_e(z,\gamma) \geqslant 4d(z,\partial D)^{-1}$, where $d(z,\partial D)$ is the euclidean distance from z to ∂D, then $\kappa_D(z,\gamma) \geqslant 2$.*

Proof. Set $G = \{w : |w-z| < d(z,\partial D)\}$. Then $\lambda_G(z)^{-1} = \frac{1}{2}d(z,\partial G) = \frac{1}{2}d(z,\partial D)$ so that by (1) and the hypothesis $\kappa_G(z,\gamma) = \lambda_G(z)^{-1}\kappa_e(z,\gamma) \geqslant 2$. Thus Corollary 1 applies and we obtain $\kappa_D(z,\gamma) \geqslant 2$. This is a sharp result: equality holds when D is a disc centered at z.

3. *Characterizations for hyperbolic curvature at least* 2

We say that γ is locally euclidean convex if, when properly oriented, $\kappa_e(z,\gamma) \geqslant 0$ for all $z \in \gamma$. A Jordan curve which is locally euclidean convex bounds a convex set. The following theorem shows that the local convexity of all images of a curve under (convex) conformal mappings is tied to the hyperbolic curvature of that curve.

THEOREM 2. (i) *The curve $f(\gamma)$ is locally euclidean convex for every conformal mapping f of D if and only if $\kappa_D(z,\gamma) \geqslant 2$ for all $z \in \gamma$. The constant* 2 *is sharp.*

(ii) *The curve $f(\gamma)$ is locally euclidean convex for every convex conformal mapping f of D (that is, $f(D)$ is convex) if and only if $\kappa_D(z,\gamma) \geqslant 1$ for all $z \in \gamma$. The constant* 1 *is sharp.*

Proof. In [2] it was observed that the coefficient inequality $|a_2| \leqslant 2$ is equivalent to the estimate $|\nabla \log \lambda_G| \leqslant 2\lambda_G$ holding for every simply connected domain G. If G is convex then this can be improved to $|\nabla \log \lambda_G| \leqslant \lambda_G$ because a normalized conformal mapping onto a convex domain satisfies $|a_2| \leqslant 1$. We use this to establish the sufficiency in (i) and (ii).

Let $a = 2$ (or 1) in case (i) (or case (ii) and let $w = f(z)$ be a conformal (convex conformal) mapping of D onto $G = f(D)$. Then (1) and conformal invariance imply that

$$\begin{aligned}\kappa_e(w, f(\gamma)) &= \lambda_G(w)\,\kappa_D(z,\gamma) + \frac{\partial}{\partial n}\log \lambda_G(w)\\ &\geqslant a\lambda_G(w) - |\nabla \log \lambda_G(w)| \geqslant 0,\end{aligned}$$

for all $z \in \gamma$.

For the necessity in case (i) we let f be the mapping of D onto $G = \mathbb{C}\backslash(-\infty, -1/4]$ which takes some fixed point $z_0 \in \gamma$ to 0 and so that the normal to $f(\gamma)$ at 0 is in the direction $-\nabla \log \lambda_G(0)$. For $f(\gamma)$ to be locally euclidean convex we must have $\kappa_e(0, f(\gamma)) \geqslant 0$ from which

$$\kappa_D(z_0, \gamma) = \kappa_G(0, f(\gamma)) = \lambda_G(0)^{-1}(\kappa_e(0, f(\gamma)) + |\nabla \log \lambda_G(0)|)$$
$$\geqslant \lambda_G(0)^{-1}|\nabla \log \lambda_G(0)| = 2.$$

A similar argument using a mapping of D onto a half plane establishes the necessity in case (ii).

To check the sharpness of the constant in (i) we consider conformal mappings f of the disc. The circle $|z| = r, 0 < r < 1$, has constant hyperbolic curvature $\kappa_B = \frac{1}{2}(r + r^{-1}) = \coth r_h$, where r_h is the hyperbolic radius. This will be at least 2 and so will be mapped to a convex curve under any conformal mapping of B precisely when $r \leqslant 2 - \sqrt{3}$; the sharp 'radius of convexity' for the class S, [3, Chapter 2].

Incidentally, case (ii) says that under a convex conformal mapping of B the image of any circle contained in B will be convex since the curvature is $\coth r_h > 1$ for such circles. The same is true for horocycles, all of which have constant hyperbolic curvature 1. This observation goes back to Study [5].

For the sharpness of the constant 1 in case (ii) let D be the upper half-plane and map D onto the strip $0 < \operatorname{Im} w < \pi$ by $f(z) = \log z$. A straight ray γ in D making an angle $\phi \in (0, \pi)$ with the real axis and traversed upward has $\kappa_D(z, \gamma) = \cos \phi$ for all $z \in \gamma$. If the endpoint of γ is on the real axis at $-x_0 < 0$, then $\kappa_e(f(z), f(\gamma)) = -\sin(\phi + \operatorname{Im} f(z))$. Regardless of ϕ, this last quantity changes sign as z traverses γ since $\operatorname{Im} f(z)$ varies from π to 0. Consequently $f(\gamma)$ will not be euclidean convex no matter how close to 1 we make $\cos \phi$.

Jordan curves in Theorem 2 may be characterized in another more geometric way. The formulation is especially appealing in the case $D = B$, where curves of constant hyperbolic curvature are circular arcs.

DEFINITION. Let $a \geqslant 0$. A closed subset E of B is *hyperbolically a-convex* if for each pair $z_1, z_2 \in E$ the two shortest arcs of constant hyperbolic curvature a from z_1 to z_2 are also contained in E.

Note that if E is hyperbolically a-convex then it is also hyperbolically a'-convex for $0 \leqslant a' \leqslant a$. Zero-convexity is the same as convexity. If $a \geqslant 1$, so the circles of constant hyperbolic curvature a lie entirely in $\bar{B}$, the definition above may be reformulated to require that $E \cap \alpha$ be connected whenever α is a circle of hyperbolic curvature a which meets E. It is not hard to show that a sufficient condition for this to happen is that $\partial E = \gamma$ be a smooth Jordan curve with $\kappa_B(z, \gamma) \geqslant a$ for all $z \in \gamma$. Now, the euclidean version of this, with $a \geqslant 0$, is easy to prove, undoubtedly in the literature, and the ideas carry over to the hyperbolic case without difficulty. We omit the details. The following theorem establishes the converse of this sufficient condition for the values $a = 1, 2$ appearing in Theorem 2.

THEOREM 3. *Let $a = 1$ or 2. If E is a closed subset of B with $\partial E = \gamma$ a smooth Jordan curve, then E is hyperbolically a-convex if and only if $\kappa_B(z, \gamma) \geqslant a$ for all $z \in \gamma$.*

Proof. For the proof of necessity, let $z_1, z_2 \in E$ and let α_1, α_2 be the two circles of constant hyperbolic curvature a through z_1 and z_2. The intersection, G, of the discs bounded by α_1 and α_2 is contained in E since E is a-convex. Consider the case $a = 2$ and let f be a conformal mapping of B. By Theorem 2 part (i) the images under f of the discs bounded by the α_i are euclidean convex so $f(G)$ is euclidean convex and the segment from $f(z_1)$ to $f(z_2)$ is in $f(E)$. Since $z_1, z_2 \in E$ were arbitrary we conclude that $f(E)$ is convex for any conformal mapping of B. By Theorem 2(i) again, it follows that $\kappa_B(z,\gamma) \geqslant 2$ for all $z \in \gamma$.

The proof for $a = 1$ is carried out in the same way using Theorem 2 part (ii). This case is equivalent to the result in [4].

4. *Concluding remarks*

Theorem 1 and its corollaries are another geometric look at the mix of local and global considerations in conformal mapping. Here it is the geodesic curvature of an arc and the simple connectivity of the domain in which the arc lies. As to generalizations, since Corollary 1, for example, is equivalent to $|a_2| \leqslant 2$ one cannot expect such a result to hold for multiply connected domains without further assumptions. Take G to the punctured unit disc $0 < |z| < 1$, D to be $|z| < 1$ and let γ be a vertical line segment through $x_0 \in (0, 1/e)$, oriented so that its normal points to the right. Then $\kappa_G(x_0, \gamma) = (\log(1/x_0) - 1) \in (0, \infty)$ while $\kappa_D(x_0, \gamma) = -x_0 < 0$.

It is possible to give some extensions of Theorem 1 *et al.* to multiply connected domains when the covering map $p: B \to D$ behaves quantitatively like a univalent function (see [2] and the references there) but a better understanding should involve arguments of a more differential geometric nature.

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