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A generalization of Nehari's univalence criterion

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This note is a sequel to our paper [OS] where we generalized the Schwarzian derivative to conformal mappings of Riemannian manifolds. There we found that many of the phenomena familiar from the classical theory have counterparts in the more general setting. Here we advance this another step by giving a generalization of the well known univalence criterion of Nehari [N]. Despite its relatively advanced age, this result continues to generate interest, see [L]. The argument used here in the general case, if specialized to the situation considered by Nehari, gives a somewhat different and a more geometric proof of his theorem than is often presented.

We want to keep this note short, since the proof of the Theorem is really quite simple, and also fairly self-contained. We shall need a number of facts from our earlier paper and we collect them here with very little additional discussion. We refer the reader to that paper for more details.

1. Background

Let M be an n-dimensional Riemannian manifold, $n \geq 2$, with metric g and Riemannian connection ∇. If $\hat{g} = e^{2\varphi}g$ is a conformal metric we define a $(0, 2)$, symmetric, traceless tensor field

$$B(\varphi) = B_g(\varphi) = \operatorname{Hess}(\varphi) - d\varphi \otimes d\varphi - \frac{1}{n}\{\Delta\varphi - \|\operatorname{grad} \varphi\|\}g,$$

where Hess is the Hessian operator. Recall that $\operatorname{Hess}(\varphi)(X, Y) = X(Y\varphi) - (\nabla_X Y)\varphi$ for a pair of vector fields X, Y. If $f : (M, g) \to (M', g')$ is a conformal local diffeomorphism between Riemannian manifolds with $f^*g' = e^{2\varphi}g$, $\varphi = \log\|df\|$, then we define the *Schwarzian tensor* of f to be $\mathcal{S}(f) = \mathcal{S}_g(f) = B_g(\varphi)$. If f is analytic on a domain in C with $f' \neq 0$ then, computing with respect to the Euclidean metric in standard coordinates, one finds

$$\mathscr{S}(f) = B(\log|f'|) = \begin{bmatrix} \operatorname{Re} S(f) & -\operatorname{Im} S(f) \\ -\operatorname{Im} S(f) & -\operatorname{Re} S(f) \end{bmatrix}, \tag{1}$$

where $S(f) = (f''/f')' - \frac{1}{2}(f''/f')^2$ is the usual Schwarzian derivative of an analytic function.

By $\|\mathscr{S}(f)\|$ we mean the norm of $\mathscr{S}(f)$, with respect to g, as a bilinear form on each tangent space, that is

$$\|\mathscr{S}(f)\| = \max\{|\mathscr{S}(f)(X, Y)| : \|X\| = \|Y\| = 1\}. \tag{2}$$

In the case of analytic functions and the Euclidean metric $\|\mathscr{S}(f)\| = |S(f)|$, from (1).

If $B_g(\varphi) = 0$ then $\hat{g} = e^{2\varphi}g$ is said to be a *Möbius metric* (with respect to g). The most general Möbius metric on $\mathbf{R}^n$ conformal to the Euclidean metric has

$$\varphi(x) = -\log(a|x|^2 + b \cdot x + c), \qquad a, c \in \mathbf{R},\ b \in \mathbf{R}^n, \tag{3}$$

where $b \cdot x$ is the Euclidean inner product. These metrics have constant curvature $4ac - |b|^2$. In particular, the Poincaré metric

$$\frac{1}{1-|z|^2}|dz|$$

on the disk (Gaussian curvature -4, scalar curvature -8) and the spherical metric

$$\frac{2}{1+|x|^2}|dx|$$

on $\mathbf{R}^n \cup \{\infty\}$ (sectional curvatures 1, scalar curvature $n(n-1)$) are Möbius metrics.

The most important property of the tensor $B_g(\varphi)$ that we use here is the way it changes when there is a conformal change in the metric g. Thus if $\hat{g} = e^{2\varphi}g$ and if σ is any smooth function on M then

$$B_g(\varphi + \sigma) = B_g(\varphi) + B_{\hat{g}}(\sigma). \tag{4}$$

This is entirely equivalent to the formula

$$\mathscr{S}_g(h \circ f) = f^*\mathscr{S}_{g'}(h) + \mathscr{S}_g(f) \tag{4$'$}$$

for the composition of two conformal local diffeomorphisms $(M, g) \xrightarrow{f} (M', g') \xrightarrow{h} (M'', g'')$. Equation (4)′ is a generalization of the well known classical formula.

One consequence of this is that if $f:(M,g)\to(M',g')$ is conformal and if $\hat{g}=e^{2\varphi}g$ is a Möbius metric (here φ is not necessarily $\log\|df\|$) then $\mathcal{S}_g(f)=\mathcal{S}_{\hat{g}}(f)$. A second consequence that we need has to do with Möbius metrics on the sphere, but before stating this we make one more general remark. The substitution $u=e^{-\varphi}$ converts the equations $B(\varphi)=0$, $B(\varphi)=p$ into the linear equations

$$\begin{aligned}&(a)\ \operatorname{Hess}(u)=\left(\frac{\Delta u}{n}\right)g\\ &(b)\ \operatorname{Hess}(u)+up=\left(\frac{\Delta u}{n}\right)g,\end{aligned}\tag{5}$$

respectively. (Note that in the second equation p is a symmetric $(0,2)$ tensor field of trace zero.) We let $\mathcal{U}(M)$ be the space of solutions to (5*a*). If $u\in\mathcal{U}(M)$ with $u>0$ then $B(-\log u)=0$.

Let S^n be the sphere and let g_0 denote the standard round metric. Using stereographic coordinates we write g_0 as $4(1+|x|^2)^{-2}|dx|^2=e^{2\varphi_0}(euc)$ on $\mathbf{R}^n\cup\{\infty\}$. Since, as mentioned, $B(\varphi_0)=0$ with respect to the Euclidean metric, we find from (3) and (4) that the general solution of $B_{g_0}(\varphi)=0$ is of the form

$$\varphi(x)=-\log\left(\frac{A|x|^2+B\cdot x+C}{|x|^2+1}\right),\qquad A,C\in\mathbf{R},\ B\in\mathbf{R}^n,$$

in these coordinates, and a general $u\in\mathcal{U}(S^n)$ is of the form

$$u(x)=\frac{A|x|^2+B\cdot x+C}{|x|^2+1}.$$

Then $u^{-2}g_0$ has curvature $AC-\frac{1}{4}|B|^2$. We see from this that if $u^{-2}g_0$ is flat then u vanishes at precisely one point in $\mathbf{R}^n\cup\{\infty\}=S^n$ and hence is otherwise of one sign. This is the fact we shall use later and we state it as follows.

LEMMA. *For each $p\in S^n$ there is a $u\in\mathcal{U}(S^n)$ such that $u(p)=0$, $u>0$ on $S^n\setminus\{p\}$ and $u^{-2}g_0$ is flat.*

We need one more formula. For a metric g on M let $k=(n(n-1))^{-1}\operatorname{scal}(g)$, where scal is the scalar curvature. If $\hat{g} = e^{2\varphi}g$ and $\hat{k}$ is the corresponding quantity then

$$\hat{k} = e^{-2\varphi}\left\{k - \frac{2}{n}\Delta\varphi - \frac{n-2}{n}\|\operatorname{grad}\varphi\|^2\right\}. \tag{6}$$

2. Injectivity criterion.

We may now state

THEOREM. *Let (M, g) be a Riemannian manifold of dimension $n \geq 2$ and $f: (M, g) \to (S^n, g_0)$ a conformal local diffeomorphism. Suppose that the scalar curvature of M is bounded above by $n(n-1)K$ for some $K \in \mathbf{R}$, and that any two points in M can be joined by a geodesic of length $< \delta$ for some $0 < \delta \leq \infty$. If*

$$\|\mathcal{S}(f)\| \leq \frac{2\pi^2}{\delta^2} - \frac{1}{2}K$$

then f is injective.

Proof. Let $\varphi = \log\|df\|$, so that $f^*g_0 = e^{2\varphi}g = \hat{g}$. Let $x \in M$, $p = f(x)$ and choose a function $u \in \mathcal{U}(S^n)$ which vanishes at p, which is otherwise positive and is such that $u^{-2}g_0$ is flat. Define

$$w = (u \circ f)e^{-\varphi}$$

on M. Then

$$w^{-2}g = f^*(u^{-2}g_0)$$

is a flat metric on $M \backslash f^{-1}(p)$. Using (4) we find that

$$\begin{aligned} B_g(-\log w) &= B_g(\varphi - \log(u \circ f)) = B_g(\varphi) + B_{\hat{g}}(-\log(u \circ f)) \\ &= \mathcal{S}(f) + f^*B_{g_0}(-\log u) = \mathcal{S}(f) \end{aligned}$$

Hence from (5b) we may write

$$\operatorname{Hess}(w) = -w\mathcal{S}(f) + \frac{\Delta w}{n}g. \tag{7}$$

(Equation (7) holds on all of M.)

Let k be $(n(n-1))^{-1}$ times the scalar curvature of g. Since the metric $w^{-2}g$ is flat, equation (6) gives

$$\begin{aligned} 0 &= k - \frac{2}{n}\Delta(-\log w) - \frac{n-2}{n}\|\text{grad}\log w\|^2 \\ &= k + \frac{2}{n}\frac{\Delta w}{w} - \frac{\|\text{grad}\, w\|^2}{w^2}. \end{aligned} \tag{8}$$

The assumption on the scalar curvature then implies

$$\frac{\Delta w}{n} \geq \frac{1}{2}w\left\{-K + \frac{\|\text{grad}\, w\|^2}{w^2}\right\}. \tag{9}$$

Now let $\gamma : [0, l) \to M$, $l \leq \delta$, be a unit speed geodesic for g with $\gamma(0) = x$. Write $w(t)$ for w evaluated along γ. Then $w(0) = 0$ and $w(t) > 0$ for small positive t. From (7), (9) and the bound on $\|\mathscr{S}(f))\|$ we obtain, whenever $w(t) > 0$,

$$\begin{aligned} w'' &= \text{Hess}\,(w)(\dot\gamma, \dot\gamma) = -w\mathscr{S}(f)(\dot\gamma, \dot\gamma) + \frac{\Delta w}{n} \\ &\geq -w\left(\frac{2\pi^2}{\delta^2} - \frac{1}{2}K\right) + \frac{1}{2}w\left(-K + \left(\frac{w'}{w}\right)^2\right) \\ &= -\frac{2\pi^2}{\delta^2}w + \frac{1}{2}\frac{(w')^2}{w}. \end{aligned}$$

We write this as

$$(w^{1/2})'' \geq -\frac{\pi^2}{\delta^2}w^{1/2}. \tag{10}$$

To summarize, $w(0) = 0$, $w(t) > 0$ for sufficiently small positive t and (10) holds whenever $w(t) > 0$. Since $l \leq \delta$, the simplest Sturm comparison theorem (see e.g. [BR, p. 23]) guarantees that $w(t)$ cannot vanish again. But then $f(\gamma(t))$, $t \in (0, l)$ cannot equal $f(x)$ and the theorem is proved.

Remark. Observe that we have actually proved a more general statement, to the effect that if the hypotheses are satisfied along a geodesic of length $< \delta$ then f is injective along that geodesic.

3. Concluding remarks

We easily deduce Nehari's original theorems as a corollary.

COROLLARY. (*Nehari*) *If f is analytic and locally injective in $|z| < 1$ and if either*

$$(a)\quad |S(f)(z)| \le \frac{2}{(1-|z|^2)^2}, \quad or$$

$$(b)\quad |S(f)(z)| \le \frac{\pi^2}{2}$$

then f is injective in $|z| < 1$.

Proof. In both cases we apply the Theorem with M the unit disk. For (b) we use the Euclidean metric, for which $\delta = 2$ and $K = 0$. The condition is then $\|S(f)\| \le \pi^2/2$, which is (b).

For (a) we use the Poincaré metric $(1 - |z|^2)^{-1}|dz|$. Then $\delta = \infty$ and $K = -4$ and the condition is $\|\mathscr{S}(f)\| \le 2$. To see that this translates to (a) we recall first that the Poincaré metric on $|z| < 1$ is a Möbius metric, so that the Schwarzian tensor is the same in either the Poincaré or the Euclidean metric. Second, the norm is with respect to the Poincaré metric, so if X is a unit tangent vector at z in the Euclidean metric then $(1 - |z|^2)X$ is a unit vector in the Poincaré metric. Thus

$$\|\mathscr{S}(f)(z)\| = (1 - |z|^2)^2 |S(f)(z)|,$$

and (a) follows at once.

Of course one can deduce other such criteria. For example, if we consider the disk with the spherical metric $2(1 + |z|^2)^{-1}|dz|$, then $\delta = \pi$ and $K = 1$. Again since this is a Möbius metric the condition $\|\mathscr{S}(f)\| \le 3/2$ translates to

$$|S(f)(z)| \le \frac{6}{(1+|z|^2)^2}$$

as a sufficient condition for injectivity. This is sharp, as shown by

$$f(z) = \left(\frac{1+iz}{1-iz}\right)^{2+\varepsilon}, \qquad \varepsilon > 0.$$

Nehari's theorems are also sharp.

We can also phrase these kinds of conditions differently by stating, for example, that if either

$$|S(f)(z)| \le 2\left(\frac{\pi^2}{\delta^2}+1\right)(1-|z|^2)^{-2}, \qquad \text{or}$$

$$|S(f)(z)| \le 2\frac{\pi^2}{\delta^2},$$

for $|z|<1$ then f is injective in any disk D in $|z|<1$ of hyperbolic or Euclidean diameter δ respectively. Similarly, if Ω is a convex domain with its Euclidean metric then

$$|S(f)(z)| \le \frac{2\pi^2}{(\operatorname{diam}\Omega)^2}$$

is a sufficient condition for injectivity. This is a sample. In his Stanford Thesis (in progress) M. Chuaqui has shown how to derive all of the known univalence criteria of this type from the general theorem.

Finally, we would like to offer one more proof of part (a) of Nehari's original theorem. It too is brief and uses an auxiliary function $u \in \mathscr{U}(S^2)$ provided by the Lemma. But it is not a differential equations argument and it is unlike any other proof of this result we know; in particular the constant 2 enters in a rather different way. Although we have not yet been able to use this reasoning in a more general setting, we feel it may be of independent interest.

Suppose, then, that f is analytic and locally injective in $|z|<1$ and satisfies

$$|S(f)(z)| \le \frac{2}{(1-|z|^2)^2}$$

there. Let z_0 be a point in the disk and choose $u \in \mathscr{U}(S^2)$ as in the Lemma with $u(f(z_0))=0$ and $u>0$ elsewhere. Let $D_r=\{|z|<r\}$, $0<r<1$ and denote by

$$g_r = \frac{r^2}{(r^2-|z|^2)^2}|dz|^2$$

the Poincaré metric for D_r. Choose $r<1$ so that $z_0 \in D_r$ and consider the conformal maps

$$(D_r, g_r) \xrightarrow{i} (D_1, g_1) \xrightarrow{f} (S^2, u^{-2}g_0),$$

where i is the inclusion map.

We follow the proof of the Theorem up to equation (8). Thus we define a flat metric on $D_r \setminus \{z_0\}$ by

$$w^{-2}g_r = (f \circ i)^*(u^{-2}g_0),$$

where we compute the conformal factor w to be

$$w = (u \circ f \circ i)\frac{1+|f|^2}{2|f'|}\frac{r}{r^2-|z|^2}. \tag{11}$$

w is defined on D_r and we want to show that it can vanish only at z_0.

We have that $\mathcal{S}_{g_r}(i) = 0$, whence by (4)′ $\mathcal{S}_{g_r}(f \circ i) = \mathcal{S}_{g_1}(f)$, and we may write (7) as

$$\text{Hess}\,(w) = -w\mathcal{S}_{g_1}(f) + \frac{\Delta w}{2}g_r. \tag{12}$$

(In (12) Hess (w) and Δw are computed with respect to g_r.) Equation (8) becomes in this case,

$$-4w^2 - \|\text{grad}\, w\|_{g_r}^2 + w\,\Delta w = 0.$$

The argument now takes a different turn. At a critical point of w which is not a zero we have $\Delta w = 4w$, so

$$\text{Hess}\,(w) = -w\mathcal{S}_{g_1}(f) + 2wg_r. \tag{13}$$

By hypothesis $\|\mathcal{S}_{g_1}(f)\|_{g_1} \le 2$ and it is easy to check that then $\|\mathcal{S}_{g_1}(f)\|_{g_r} < 2$. Therefore Hess (w) is positive definite at such a critical point, and the point is a local minimum. On the other hand w a positive except when it is zero, hence *all* critical points are minima.

Now from (11), $w \to \infty$ as $|z| \to r$, so on a slightly smaller disk grad w gives an outward pointing vector field along the boundary. Hence the sum of the indices of the vector field grad w at the critical points in D_r is 1, the Euler characteristic of a closed disk. (see, e.g. [H]). But each critical point is a local minimum and so has index 1. Therefore there is exactly one critical point, z_0, the point where w vanishes. We now let $r \to 1$ and the result is proved.

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