Bézier Parabolas and Medians of a Triangle

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Bézier curves were introduced in the 1970's, by Pierre Bézier, for CAD/CAM applications. They have since found many uses, perhaps most famously in the Adobe PostScript language and in Adobe Illustrator. Medians of triangles were introduced earlier. They are used most famously to locate the centroid of a triangle.

In general, an n'th degree Bézier curve depends on $n+1$ 'control points' and can be defined recursively by taking convex combinations of pairs of Bezier curves of lower degree, starting at points (0'th degree). The Bézier curves one sees in most applications are cubics, with four control points; your students will study them in any course on computer graphics. Our purpose here is to observe some nice connections between Bézier *parabolas* and the medians of the triangle determined by their three control points.

Fix three noncolinear points P_0, P_1 and P_2 in $\mathbf{R}^2$ and let i,j,k stand for indices drawn from $\{0, 1, 2\}$, distinct when more than one index appears in a formula. We define parametric curves of degree 0, 1, and 2, with a parameter t, $0 \le t \le 1$, by

$$\begin{aligned} p_i(t) &= P_i \\ p_{ij}(t) &= (1-t)p_i(t) + tp_j(t) = (1-t)P_i + tP_j \\ p_{ijk}(t) &= (1-t)p_{ij}(t) + tp_{jk}(t) = (1-t)^2P_i + 2t(1-t)P_j + t^2P_k \end{aligned}$$

The p_{ijk} are *Bézier parabolas with control points* P_i, P_j and P_k (in that order). Since $p_{ijk}(t) = p_{kji}(1-t)$ there are, geometrically, only three curves. The curve p_{ijk} has endpoints P_i and P_k and is tangent to p_{ij} and to p_{jk} at P_i and P_k, respectively. The point P_j is not on p_{ijk}; moving P_j 'controls' the shape of the curve while pinning it at the endpoints and maintaining the tangencies there. Each point on p_{ijk} is a convex combination of the control points, so the curve lies within the triangle with vertices P_0, P_1, and P_2.

Here are two pictures showing Bézier parabolas in the triangles determined by their three control points. If you draw the medians of the triangles you will see that they pass through the points where the parabolas intersect. What you see is what really happens, and then some.

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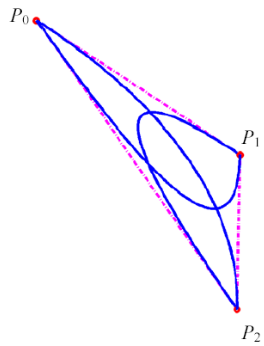

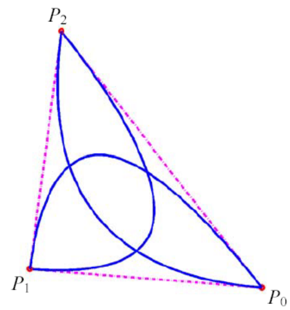

Solving the equations simultaneously to find the intersection points P_0^1, P_1^1 and P_2^1 of the Bézier parabolas is not recommended. (The superscript 1 is meant to suggest 'first generation'. There will be later generations.) Instead we find simply, and remarkably, that the points are *parametrically* 1/3 and 2/3 along the way:

$$\begin{aligned} p_{120}(1/3) &= p_{012}(2/3) = \frac{1}{9}P_0 + \frac{4}{9}P_1 + \frac{4}{9}P_2; \quad \text{this is } P_0^1. \\ p_{201}(1/3) &= p_{120}(2/3) = \frac{4}{9}P_0 + \frac{1}{9}P_1 + \frac{4}{9}P_2; \quad \text{this is } P_1^1. \\ p_{012}(1/3) &= p_{201}(2/3) = \frac{4}{9}P_0 + \frac{4}{9}P_1 + \frac{1}{9}P_2; \quad \text{this is } P_2^1. \end{aligned}$$

Now take, for example, the median from P_0, which is given by the vector $(1/2)(P_1+P_2)-P_0$. The vector from P_0 to P_0^1 is

$$P_0^1 - P_0 = \frac{4}{9}P_1 + \frac{4}{9}P_2 - \frac{8}{9}P_0 = \frac{8}{9}\left(\frac{1}{2}P_1 + \frac{1}{2}P_1 - P_0\right),$$

so not only is P_0^1 along the median from P_0, it's exactly 8/9 of the way along. Similar results hold for the other medians and the points of intersection of the other Bézier parabolas.

Being a mere 1/9 away from the actual midpoints of the sides of the triangle, one might be tempted to refer to the intersections of the Bézier parabolas as pseudo-midpoints of the triangle. Here's another reason for the name. If we look at the vector from P_0^1, to P_1^1, for example again, we find

$$P_1^1 - P_0^1 = \left(\frac{4}{9}P_0 + \frac{1}{9}P_1 + \frac{4}{9}P_2\right) - \left(\frac{1}{9}P_0 + \frac{4}{9}P_1 + \frac{4}{9}P_2\right) = \frac{1}{3}(P_0 - P_1),$$

and again analogous results hold for the other points. Thus, briefly: The segment joining two pseudo-midpoints of a triangle is parallel to the third side and 1/3 as long. Moreover, the triangle with vertices at the pseudo-midpoints is similar to the original triangle, with lengths scaled by 1/3. Compare these with Euclid's theorems on the segments joining the actual midpoints and the triangle so formed.

What else about the new littler triangle made from the pseudo-midpoints? Its centroid is

$$\begin{aligned}\frac{1}{3}\left(P_0^1+P_1^1+P_2^1\right) &= \frac{1}{3}\left(\left(\frac{1}{9}P_0+\frac{4}{9}P_1+\frac{4}{9}P_2\right)+\left(\frac{4}{9}P_0+\frac{1}{9}P_1+\frac{4}{9}P_2\right)+\left(\frac{4}{9}P_0+\frac{4}{9}P_1+\frac{1}{9}P_2\right)\right)\\ &= \frac{1}{3}\left(P_0+P_1+P_2\right),\end{aligned}$$

the same as the centroid of the original triangle (which in turn is also the centroid of the triangle made from the actual midpoints). Because of this, if we use P_0^1, P_1^1 and P_2^1 as control points for new Bézier parabolas then their intersections, P_0^2, P_1^2, P_2^2, lie on the medians of the original triangle. And so on. Here is a picture showing a few generations of Bézier parabolas and their intersections.