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Results in Mathematics

On the coefficients of small univalent functions

M. Chuaqui B. Osgood Ch. Pommerenke *)

Abstract: For every $\alpha > 0$ there exists an analytic univalent function $f(z) = a_1 z + a_2 z^2 + \dots$ satisfying

$$(1 - |z|^2)\,|f''(z)/f'(z)| \leq \alpha \quad \text{for } |z| < 1$$

such that $|a_n| > n^{c\alpha^2 - 1}$ for infinitely many n.

1. Introduction

Let the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

be analytic in the unit disk $\mathbb{D}$ and let $f'(z) \neq 0$. We assume that

$$(1 - |z|^2)|f''(z)/f'(z)| \leq \alpha \ \text{ for } \ z \in \mathbb{D}. \tag{1.2}$$

If f is univalent then (1.2) holds with $\alpha = 6$ [7, Prop. 1.2]. Conversely, if (1.2) holds with $\alpha \leq 1$ then f is univalent, and if $\alpha < 1$ then $f(\mathbb{D})$ is a quasidisk [1].

A related condition on the Schwarzian derivative S_f is

$$||S_f|| := \sup\{(1 - |z|^2)^2 |S_f(z)| : z \in \mathbb{D}\} \leq \alpha. \tag{1.3}$$

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If f is univalent then $||S_f|| \leq 6$. Conversely, if $||S_f|| \leq 2$ then f is univalent [6]; this Nehari class has been studied e.g. in [3], [4]. If $f''(0) = 0$ then

$$||S_f|| \leq \alpha \leq 2 \;\Rightarrow\; (1.2) \;\Rightarrow\; ||S_f|| \leq \text{const} \cdot \alpha\,. \tag{1.4}$$

We shall study univalent functions that are small in the sense that $\alpha > 0$ is small in (1.2) or equivalently in (1.3); see (1.4). This does not imply anything about the regularity of the boundary. Indeed there are functions satisfying (1.2) with arbitrarily small α such that $\partial f(\mathbb{D})$ does not possess a tangent at any point [7, p.190 and p.193/194].

We need a result of Makarov [5]; explicit bounds were given by Rohde [7, p.191].

Proposition 1. *For small $\alpha > 0$ there exists a univalent function $g(z) = z + \ldots$ such that*

$$(1 - |z|^2)\, |g''(z)/g'(z)| \leq \alpha/2 \quad \textit{for} \quad z \in \mathbb{D} \tag{1.5}$$

and, with some constant $c_0 > 0$,

$$\int_{-\pi}^{\pi} |g'(re^{it})|\, dt > (1 - r)^{-c_0\alpha^2} \quad \textit{for} \quad r_0 < r < 1. \tag{1.6}$$

In order to obtain g from the function f constructed by Makarov we set $g' = (f')^{\alpha/12}$.

The standard way to obtain an upper bound of the coefficients is to use the elementary estimate

$$|n\, a_n| < \text{const} \int_{-\pi}^{\pi} |f'(re^{it})| dt, \quad r = 1 - \frac{\text{const}}{n}. \tag{1.7}$$

Carleson and Jones have shown that, surprisingly, one does not lose much in (1.7); see [2, Th.2] and Proposition 2. below.

If f satisfies (1.2) then [7, Exer. 8.3.4] and (1.7) show that

$$a_n = O\left(n^{\alpha^2/4 - 1}\right) \qquad (n \to \infty). \tag{1.8}$$

We shall prove that this estimate is best possible except for the constant.

Theorem . *For sufficiently small $\alpha > 0$, there exists a univalent function f with*

$$(1-|z|^2)\,|f''(z)/f'(z)| \le \alpha \quad \text{for} \quad z \in \mathbb{D}$$

such that, with some constant $c > 0$,

$$|a_n| > n^{c\alpha^2 - 1} \qquad \text{for infinitely many } n. \tag{1.9}$$

2. The Carleson-Jones modification

We shall need the following variant of an important theorem of Carleson and Jones [2, Th.1]. Our variant contains information about ψ''/ψ'.

Proposition 2. *Let $n > 1600$ and $\varepsilon > 0$ be given and let the analytic function $\varphi(z) = \sum \alpha_k z^k$ satisfy*

$$(1-|z|^2)\,|\varphi''(z)/\varphi'(z)| \le \gamma \le 6 \ \text{ for } z \in \mathbb{D}. \tag{2.1}$$

Then there exists a function $\psi(z) = \sum \beta_k z^k$ such that, with $r = 1 - 1600/n$,

$$|\beta_k| \ge |\alpha_k| r^{k-1} \quad \text{for } 1 \le k < n, \tag{2.2}$$

$$(1-|z|^2)|\psi''(z)/\psi'(z)| < \gamma + \varepsilon \quad \text{for } z \in \mathbb{D}, \tag{2.3}$$

$$|2n\beta_{2n}| > c'\varepsilon \int_{-\pi}^{\pi} |\varphi'(re^{it})|dt \tag{2.4}$$

where $c' > 0$ is an absolute constant.

Proof. Let $\delta > 0$. We consider the polynomial

$$p(z) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\sum_{\nu=0}^{2n}\left(1 - \frac{|\nu - n|}{n+1}\right) e^{i(n-\nu)t}\frac{|\varphi'(re^{it})|}{\varphi'(re^{it})} z^{n+\nu}dt \tag{2.5}$$

of degree $3n$ and define

$$\psi(z) = r^{-1}\varphi(rz) + \frac{\delta}{n}p(z)\varphi'(rz) \qquad (z \in \mathbb{D}). \tag{2.6}$$

Then (2.2) holds because $p(z)$ contains no powers z^k with $k < n$. Using the properties of the Fejér kernel [8, p. I 88], Carleson and Jones [2, p.178] have shown that

$$|\beta_{2n}| \geq \frac{\delta}{4\pi n} \int_{-\pi}^{\pi} |\varphi'(re^{it})| dt - |\alpha_{2n}| \geq \frac{\delta}{8\pi n} \int_{-\pi}^{\pi} |\varphi'(re^{it})| dt; \tag{2.7}$$

if the second inequality is false then we simply choose $\psi = \varphi$ instead of (2.6) and (2.2), (2.3) and (2.4) are trivially true. Furthermore

$$|p(z)| \leq 1,\ |p'(z)| \leq 3n \ \text{ for } \ z \in \mathbb{D}; \tag{2.8}$$

the first estimate follows from (2.5) and then the second from Bernstein's inequality [8, p. II 11].

Now we estimate ψ''/ψ'. Let $c_1, \dots$ denote suitable absolute constants. It follows from (2.6) that, for $z \in \overline{\mathbb{D}}$,

$$\begin{aligned} |\psi'(z)| &\geq |\varphi'(rz)| - \tfrac{\delta}{n} |p'(z)\varphi'(rz) + p(z)\, r\, \varphi''(rz)| \\ &\geq |\varphi'(rz)| \left(1 - \tfrac{\delta}{n}\left(3n + \tfrac{\gamma}{1-r^2}\right)\right) \geq |\varphi'(rz)|(1 - c_1\delta) \end{aligned} \tag{2.9}$$

by (2.8) and (2.1). Furthermore

$$|\psi''(z)| \leq |\varphi''(rz)| + \frac{\delta}{n} \left| p''(z)\varphi'(rz) + 2r\, p'(z)\varphi''(rz) + r^2\, p(z)\varphi'''(rz) \right|. \tag{2.10}$$

Using a standard argument we deduce from (2.8) and from (2.1) that

$$|p''(z)| \leq \frac{c_3 n}{1-|z|}, \ \left| \frac{\varphi'''(rz)}{\varphi'(rz)} \right| \leq \frac{c_4 \gamma}{(1-r)(1-|rz|)} \leq \frac{c_5 \gamma n}{1-|z|}.$$

Hence we obtain from (2.10) by (2.1) and (2.8) that

$$(1-|z|^2) \left| \frac{\psi''(z)}{\varphi'(rz)} \right| \leq \frac{\gamma + 2\delta(c_3 + 3\gamma + c_5\gamma)}{1 - c_1\delta} \leq \gamma + c_6\delta.$$

Therefore (2.3) holds if we choose $\delta = \varepsilon / c_6$, and (2.4) holds by (2.7).

3. Proof of the theorem

(a) By $c_1, \ldots$ we denote positive absolute constants. Let $0 < \alpha < 1$ and let g be the function in Proposition 1 of Makarov. Let $0 < q_k < 1$ and let m_k be a (large) integer. We define h_k by

$$h_k'(z) = q_k\, g'(z^{m_k}) \quad (z \in \mathbb{D}), \qquad h_k(0) = 0. \tag{3.1}$$

Let $|z| = r < 1$ and write $m = m_k$. We obtain from (3.1) and (1.5) that

$$(1-r^2)\left|\frac{h_k''(z)}{h_k'(z)}\right| = (1-r^2)m\, r^{m-1}\left|\frac{g''(z^m)}{g'(z^m)}\right| \le \frac{\alpha m\, r^{m-1}(1-r^2)}{2(1-r^{2m})} < \frac{\alpha}{2}. \tag{3.2}$$

If $r \le 1 - 1/\sqrt{m}$ then $r^m \le (1-1/\sqrt{m})^m < e^{-\sqrt{m}}$ and thus

$$(1-r^2)\,|h_k''(z)/h_k'(z)| < e^{-\sqrt{m}/2} \tag{3.3}$$

provided that $m = m_k$ is large.

It follows from (1.5) by integration that

$$|g'(z)| \le c_1(1-r)^{-1/2} \tag{3.4}$$

and thus by the maximum principle that $|z^{-1}(g'(z)-1)| \le c_2(1-r)^{-1/2}$. Hence, by (3.1),

$$|h_k(z) - q_k z| = q_k\left|\int_0^z [g'(\zeta^m) - 1]d\zeta\right| \le c_2 \int_0^1 s^m(1-s^m)^{-1/2}ds \le \frac{c_3}{m}.$$

Choosing $q_k = 1 - c_3\, m_k^{-1} - m_k^{-1/8}$ we deduce that

$$|h_k(z)| < q_k + c_3\, m_k^{-1} = 1 - m_k^{-1/8} \quad \text{for } z \in \mathbb{D}. \tag{3.5}$$

(b) We will recursively define integers m_k and n_k with

$$m_k > \max\left[n_k, \left(c_4\, \alpha^{-1}\, 2^{k+2}\right)^8\right] \quad \text{for } k = 1, 2, \ldots \tag{3.6}$$

(see (3.10) below for c_4) and functions

$$f_k(z) = z + \sum_{n=2}^{\infty} a_{kn} z^n \qquad (z \in \mathbb{D}) \tag{3.7}$$

starting with $f_0(z) = z$. We write

$$\eta_k = \max\left(\frac{\alpha}{2}, \sup_{z\in\mathbb{D}} (1-|z|^2)\,|f_k''(z)/f_k'(z)|\right). \tag{3.8}$$

Suppose that n_j and f_j have already been constructed for $j \le k$, also m_j for $j < k$. Let m_k satisfy (3.6) and define $\varphi_k = f_k \circ h_k$. Then

$$\frac{\varphi_k''(z)}{\varphi_k'(z)} = \frac{h_k''(z)}{h_k'(z)} + h_k'(z)\frac{f_k''(h_k(z))}{f_k'(h_k(z))} \qquad (z \in \mathbb{D})$$

and thus, for $|z| = r < 1$,

$$(1-r^2)\left|\frac{\varphi_k''}{\varphi_k'}\right| \le (1-r^2)\left|\frac{h_k''}{h_k'}\right| + \frac{(1-r^2)|h_k'|}{1-|h_k|^2}\,(1-|h_k|^2)\left|\frac{f_k''(h_k)}{f_k'(h_k)}\right|. \tag{3.9}$$

First suppose that $0 \le r \le 1 - 1/\sqrt{m_k}$. Since $h_k(\mathbb{D}) \subset \mathbb{D}$ we have $(1-r^2)|h_k'|/(1-|h_k|^2) \le 1$ and thus, by (3.9), (3.3) and (3.8),

$$(1-r^2)\left|\frac{\varphi_k''}{\varphi_k'}\right| \le e^{-\sqrt{m_k}/2} + \eta_k.$$

Now suppose that $1 - 1/\sqrt{m_k} < r < 1$. Then, by (3.9), (3.2), (3.4) and (3.5),

$$(1-r^2)\left|\frac{\varphi_k''}{\varphi_k'}\right| < \frac{\alpha}{2} + \frac{c_1(1-r^2)m_k^{1/8}}{(1-r^{m_k})^{1/2}}\,\eta_k \le \frac{\alpha}{2} + c_4\, m_k^{-1/8}\eta_k. \tag{3.10}$$

If m_k is sufficiently large we therefore obtain, by (3.8) and (3.6),

$$(1-r^2)\,|\varphi_k''(z)/\varphi_k'(z)| < \eta_k + \alpha\, 2^{-k-2} \quad \text{for} \quad |z| = r < 1. \tag{3.11}$$

This finally determines m_k.

Now let f_{k+1} be the Carleson-Jones modification of φ_k with $\varepsilon = \alpha\, 2^{-k-2}$; see Proposition 2. Since $\eta_0 = \alpha/2$ we obtain from (3.8) and (3.11) that, for $z \in I\!D$,

$$(1-|z|^2)\left|\frac{f''_{k+1}(z)}{f'_{k+1}(z)}\right| \le \eta_{k+1} < \frac{\alpha}{2} + 2\sum_{j=0}^{k} \alpha\, 2^{-j-2} < \alpha. \tag{3.12}$$

Finally we apply Proposition 1. We choose $n_{k+1} > 2n_k$ so large that (see (3.7), (2.4) and (1.6))

$$\begin{aligned} \left|n_{k+1}\, a_{k+1,n_{k+1}}\right| &> \frac{c'\alpha}{2^{k+2}} \int_{-\pi}^{\pi} \left|\varphi'_k\left(\left(1-\frac{3200}{n_{k+1}}\right)e^{it}\right)\right| dt \\ &> \frac{c_5\alpha}{2^{k+2}}\, n_{k+1}^{\;c_0\alpha^2} > n_{k+1}^{\;c_0\alpha^2/2}. \end{aligned} \tag{3.13}$$

This concludes our recursive construction.

(c) Since $m_k > n_k$ by (3.6), it follows from (3.1) and (3.7) that

$$\varphi_k(z) = f_k(h_k(z)) = \sum_{n=1}^{n_k} q_k^n\, a_{kn} z^n + O\left(z^{n_k+1}\right) \qquad (z \to 0).$$

The coefficients of the Carleson-Jones modification f_{k+1} therefore satisfy

$$|a_{k+1,n}| \ge q_k^n\, |a_{k,n}| \left(1 - \frac{3200}{n_{k+1}}\right)^{n-1} \quad \text{for } 1 \le n \le n_k$$

by (2.2). Using that $q_k = 1 - c_3\, m_k^{-1} - m_k^{-1/8} > 1 - c_6\, 2^{-k}$ by (3.6), we therefore obtain

$$|a_{k+1,n}| \ge |a_{j,n}| \prod_{\nu=j}^{k} \left[\left(1-\frac{c_6}{2^\nu}\right)^n \left(1-\frac{3200}{n_{\nu+1}}\right)^{n-1}\right]$$

for $k \ge j$ and $n \le n_j$. Hence, by (3.13) for $k = j-1$,

$$\left|a_{k+1,n_j}\right| > c_7\, n_j^{\,c_0\alpha^2/2-1} \quad \text{for } k \ge j \tag{3.14}$$

because $n_{\nu+1} > 2n_\nu$.

We select a convergent subsequence from (f_k). Its limit f satisfies (1.2) by (3.12) and satisfies (1.9) by (3.14).

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M. Chuaqui
Facultad de Matemáticas
P. Universidad Católica
Casilla 306
Santiago 22
Chile

B. Osgood
Department of Mathematics
Stanford University
Stanford, CA 94305
U.S.A.

Ch. Pommerenke
Fachbereich Mathematik
Technische Universität Berlin
Straße des 17. Juni 135
Germany

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