Design of Inline Amplifier Gains and Spacings to Minimize the Phase Noise in Optical Transmission Systems

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***Abstract*—In optical fiber transmission systems using inline amplifiers, the interaction of a signal and an amplifier noise through the Kerr effect leads to a nonlinear phase noise that can impair detection of phase-modulated signals. The authors show how to minimize the variance of the total phase noise (linear plus nonlinear) by a choice of the number of inline amplifiers N and their spacings and gains, assuming a fixed total system length L and an overall compensation of the fiber loss. In the case of a uniform amplifier spacing and a per-span loss compensation, there exists a finite N that minimizes the total phase noise. This contrasts with the well-known observation that a linear phase noise alone is minimized by a choice of an infinite N. Relaxing the constraints of the uniform spacing and/or the per-span loss compensation leads to further reduction of the total phase noise. The optimization of the spacings and the gains can be approximately formulated as a convex problem. In typical terrestrial and transoceanic systems, the total-phase-noise variance can be reduced by up to 45% and 83%.**

***Index Terms*—Nonlinear optics, optical fiber amplifiers, optical fiber communication, phase noise.**

I. Introduction

OPTICAL FIBER transmission systems using coherent or differentially coherent detection of phase-modulated signals, such as phase-shift keying (PSK), differential phase-shift keying (DPSK), or quadrature-amplitude modulation (QAM), are subject to impairment by a phase noise. An amplified spontaneous emission (ASE) from inline amplifiers is a major source of phase noise and is referred to as a linear phase noise in this paper. Gordon and Mollenauer [1] showed that a signal and an ASE can interact via the fiber Kerr nonlinearity to produce a nonlinear phase noise. Considering both linear and nonlinear phase noises, they showed that the system performance is optimized when the variance of both the phase noises are equal, corresponding to a total nonlinear phase shift of about 1 rad. We note that the nonlinear phase noise is present even in a dispersion-free fiber and should be distinguished from the Gordon–Haus jitter described in the soliton literature, wherein an ASE-induced frequency noise in a dispersive fiber causes a jitter in the arrival times of soliton pulses. In this paper, we consider only the self-phase modulation-induced nonlinear phase noise in PSK or DPSK systems, which are currently under active research [2]–[4]. Semiconductor lasers represent another important source of phase noise, but the laser phase noise is not considered in this paper. The impact of fiber nonlinearities in optical amplifier chains on the overall system performance have been investigated since the 1990s [5], [6]. Various approaches have been employed to mitigate the effect of nonlinearities, such as path-averaged signal-power minimization [7] in systems with given amplifier spacings via the choice of the amplifier gains or the management of dispersion and signal power [8], [9]. Inline phase-noise compensation techniques, such as placement of phase modulators along the fiber link [10], have been proposed. Receiver-based detection or compensation techniques have been proposed, including a nonlinear MAP/mmse detector [11] and a compensator that uses the received intensity to compensate the received phase [12]–[14], exploiting the correlation between received intensity and phase.

In m-ary PSK or DPSK lightwave systems, the phase-noise statistics completely characterize the system performance, and the variance of the total phase noise (linear plus nonlinear) is a good predictor of the system performance [1], [12]–[14]. In this paper, we study the minimization of the variance of the total phase noise through a design of amplifier gains and spacings in a link with a fixed total length L. We first show that for N amplifiers uniformly spaced along the link, there exist a finite optimal value of N for which the variance of nonlinear phase noise σ_{NL}^2 is minimized. Such a behavior is in contrast to that of the linear phase noise, as the linear-phase-noise variance σ_{L}^2 decreases monotonically toward a limit as N increases to infinity. Consequently, there also exists an optimal value of N that minimizes the combined variance of linear and nonlinear phase noises. In addition, we show that if the constraints of the uniform amplifier spacing and the per-span loss compensation are relaxed, we show that the combined phase noise can be further minimized by an appropriate design of the amplifier gains and spacings. We consider three separate cases: 1) amplifier-spacing optimization with a per-span loss compensation; 2) amplifier gain optimization for a given amplifier spacing; and 3) joint amplifier spacing and gain optimization. The combined variance $\sigma_{\mathrm{NL}}^2 + \sigma_{\mathrm{L}}^2$ can be approximated as a convex function with respect to the amplifier spacings and gains; hence, there exists a unique set of optimal amplifier spacings and gains that

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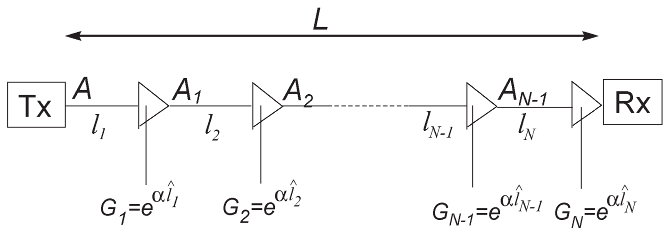

Fig. 1. Typical optical transmission system with a fixed length L and N optical amplifiers to compensate for the signal loss.

minimizes $\sigma_{\mathrm{NL}}^2 + \sigma_{\mathrm{L}}^2$. Examples show that reductions of 45% and 83% can be achieved in typical terrestrial and transoceanic systems, respectively.

II. Variance of Linear and Nonlinear Phase Noises

Consider a lightwave system shown in Fig. 1 consisting of a fiber link of fixed length L with N amplifiers distributed along the fiber to compensate for the signal loss. We assume that dispersion and multichannel effects are negligible in the system. Let A be the original field amplitude of the transmitted signal and α be the attenuation coefficient of the fiber. At the ith amplifier, let G_i and A_i and be the gain and the signal amplitude at its output, and let l_i be the spacing between ith amplifier and the preceding amplifier. If we express the gain as $G_i = \mathrm{e}^{\alpha \hat{l}_i}$, then

$$A_i = A \prod_{j=1}^{i} \mathrm{e}^{\frac{1}{2}\alpha(\hat{l}_j - l_j)}.$$

We refer to the set $\{\hat{l}_1, \hat{l}_2, \ldots, \hat{l}_N\}$ as the set of virtual spacings. Note that if $l_i = \hat{l}_i$, each amplifier exactly compensates for the signal loss from the previous span. Since the set of virtual spacings completely characterizes the amplifier gains in the system, designing the amplifier gains for phase-noise mitigation will be referred to as virtual spacing design. In this paper, the overall gain of the amplifiers is chosen to exactly compensate for the signal loss throughout the fiber, i.e., $A_N = A$. As a result, for a fixed fiber length L with N inline amplifiers, we have the constraints

$$\sum_{i=1}^{N} l_i = L \quad \text{and} \quad \sum_{i=1}^{N} \hat{l}_i = L. \tag{1}$$

Note that a common amplifier gain design is to exactly compensate the signal loss on a per-span basis. In this case, we have $\hat{l}_i = l_i$ for all $i = 1, 1, \ldots, N$. For the rest of this paper, such an approach will be referred to as a per-span compensation while overall loss compensation refers to a virtual spacing design that satisfies the constraints in (1) but does not necessarily compensate for the loss on a per-span basis. At the output of the Nth amplifier, the received field of the signal plus the noise is given by

$$E = A + \frac{A}{A_1} n_1 + \frac{A}{A_2} n_2 + \cdots + n_N \tag{2}$$

where $n_i = x_i + jy_i$, $i = 1, 2, \ldots, N$, are the noises introduced by the amplifiers, which are independent identically distributed (i.i.d.) complex zero-mean circular Gaussian random variables, i.e., $(x_i, y_i) \sim \mathcal{N}(0, \sigma_i^2 \mathcal{I})$.

The variances σ_i^2 for two polarizations are given in [15] as

$$\sigma_i^2 = 2S_{\mathrm{sp}}\Delta\nu_{\mathrm{opt}} = 2h\nu n_{\mathrm{sp}}\Delta\nu_{\mathrm{opt}}(\mathrm{e}^{\alpha\hat{l}_i} - 1) = b(\mathrm{e}^{\alpha\hat{l}_i} - 1) \tag{3}$$

where $\Delta\nu_{\mathrm{opt}}$ is the bandwidth of an optical filter at the receiver, n_{sp} is the spontaneous emission factor, and ν is the signal frequency. At high signal-to-noise ratios, the variance of the linear phase noise can be approximated as [16]

$$\sigma_{\mathrm{L}}^2 = \frac{1}{2\mathrm{SNR}} = \frac{\sum_{i=1}^{N} \sigma_i^2 \prod_{j=i+1}^{N} \mathrm{e}^{\alpha(\hat{l}_j - l_j)}}{2A^2} = \frac{1}{2}\mathbf{A\Theta} \tag{4}$$

where $\mathbf{A} = [1/A_1^2 \; 1/A_2^2 \; \cdots 1/A_N^2]$ and $\mathbf{\Theta} = [\sigma_1^2 \; \sigma_2^2 \; \cdots \; \sigma_N^2]^{\mathrm{T}}$. On the other hand, the total nonlinear phase noise at the receiver is given by [17]

$$\begin{aligned}\phi_{\mathrm{NL}} &= \gamma \Bigg[L_{\mathrm{e},1}|A_1 + n_1|^2 + L_{\mathrm{e},2}\left| A_2 + \frac{A_2}{A_1}n_1 + n_2 \right|^2 + \cdots \\ &\qquad + L_{\mathrm{e},N}\left| A_N + \frac{A_N}{A_1}n_1 + \frac{A_N}{A_2}n_2 + \cdots + n_N \right|^2 \Bigg] \\ &= \gamma \sum_{i=1}^{N} L_{\mathrm{e},i} \left| A_i + \sum_{j=1}^{i} \frac{A_i}{A_j} n_j \right|^2\end{aligned} \tag{5}$$

where γ is the nonlinear coupling coefficient. The quantity

$$L_{\mathrm{e},i} = \frac{1 - \mathrm{e}^{-\alpha l_i}}{\alpha}$$

is the effective interaction length [15]. If we let $\mathbf{x} = [x_1 \; x_2 \; \cdots \; x_N]$ and $\mathbf{y} = [y_1 \; y_2 \; \cdots \; y_N]$, the total nonlinear phase noise can be rewritten as

$$\begin{aligned}\phi_{\mathrm{NL}} &= \gamma \sum_{i=1}^{N} L_{\mathrm{e},i} \left[\left(A_i + \sum_{j=1}^{i} \frac{A_i}{A_j} x_j \right)^2 + \sum_{j=1}^{i} \frac{A_i}{A_j} y_j^2 \right] \\ &= \gamma \left(\underbrace{\sum_{i=1}^{N} L_{\mathrm{e},i}A_i^2 + 2\mathbf{w}^{\mathrm{T}}\mathbf{x} + \mathbf{x}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}\mathbf{x}}_{\varphi_1} + \underbrace{\mathbf{y}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}\mathbf{y}}_{\varphi_2} \right) \\ &= \gamma(\varphi_1 + \varphi_2)\end{aligned} \tag{6}$$

where

$$\mathbf{w}_i = \sum_{j=i}^{N} L_{\mathrm{e},j} \frac{A_j^2}{A_i}$$

and

$$\mathbf{M}=\begin{bmatrix} \sqrt{L_{e,1}} & 0 & 0 & \cdots & 0 \\ \sqrt{L_{e,2}\frac{A_2}{A_1}} & \sqrt{L_{e,2}} & 0 & \cdots & 0 \\ \sqrt{L_{e,3}\frac{A_3}{A_1}} & \sqrt{L_{e,3}\frac{A_3}{A_2}} & \sqrt{L_{e,3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{L_{e,N}\frac{A_N}{A_1}} & \sqrt{L_{e,N}\frac{A_N}{A_2}} & \sqrt{L_{e,N}\frac{A_N}{A_3}} & \cdots & \sqrt{L_{e,N}} \end{bmatrix}.$$

To obtain the variance of the nonlinear phase noise, we evaluate

$$\sigma_{\rm NL}^2 = E[\phi_{\rm NL}^2] - E[\phi_{\rm NL}]^2. \tag{7}$$

Then, if we let $C = \mathbf{M}^{\rm T}\mathbf{M}$

$$\begin{aligned}(\mathbf{y}^{\rm T}C\mathbf{y})^2 &= \left(\sum_{i=1}^{N} c_{ii}y_i^2 + 2\sum_{i,j,i>j} c_{ij}y_iy_j\right)^2 \\ &= (f(\mathbf{y}) + g(\mathbf{y}))^2 \end{aligned} \tag{8}$$

where $f(\mathbf{y}) = \sum_{i=1}^{N} c_{ii}y_i^2$ and $g(\mathbf{y}) = 2\sum_{i,j,i>j} c_{ij}y_iy_j$. Since the y_is are independent Gaussian random variables

$$E\left[y_i^p y_j^q\right] = E\left[y_i^p\right] E\left[y_j^q\right] = 0$$

when one or more of p and q are odd. Using this observation, we deduce that $E[f(\mathbf{y})g(\mathbf{y})] = 0$. Furthermore, the cross terms within $g^2(\mathbf{y})$ will vanish under expectation as well. Using the fact that $E[y_i^4] = 3\sigma_i^4$ for a Gaussian random variable, we obtain

$$\begin{aligned} E&\left[(\mathbf{y}^{\rm T}C\mathbf{y})^2\right] \\ &= E\left[f^2(\mathbf{y}) + g^2(\mathbf{y})\right] \\ &= E\left[\sum_{i=1}^{N} c_{ii}^2 y_i^4 + 2\sum_{i,j,i>j} c_{ii}c_{jj}y_i^2y_j^2 + 4\sum_{i,j,i>j} c_{ij}^2 y_i^2 y_j^2\right] \\ &= 3\sum_{i=1}^{N} c_{ii}^2\sigma_i^4 + 2\sum_{i,j,i>j} c_{ii}c_{jj}\sigma_i^2\sigma_j^2 + 4\sum_{i,j,i>j} c_{ij}^2\sigma_i^2\sigma_j^2. \end{aligned} \tag{9}$$

In addition

$$\begin{aligned} E[\mathbf{y}^{\rm T}C\mathbf{y}]^2 &= \left(E\left[\sum_{i=1}^{N} c_{ii}y_i^2 + 2\sum_{i,j,i>j} c_{ij}y_iy_j\right]\right)^2 \\ &= \left(\sum_{i=1}^{N} c_{ii}\sigma_i^2\right)^2 \\ &= \sum_{i=1}^{N} c_{ii}^2\sigma_i^4 + 2\sum_{i,j,i>j} c_{ii}c_{jj}\sigma_i^2\sigma_j^2. \end{aligned} \tag{10}$$

TABLE I

PARAMETER VALUES USED IN EVALUATING $\sigma_{\rm NL}^2, \sigma_{\rm L}^2$

P_t	1 mW (0 dBm)
α	0.25 dB/km
$\Delta\nu_{opt}$	10 GHz
n_{sp}	1.41
$\lambda = c/\nu$	1.55 μm
γ	1.2 W^{-1}/km

The variance of φ_2 is then given by

$$\begin{aligned} \sigma_{\varphi_2}^2 &= E\left[(\mathbf{y}^{\rm T}C\mathbf{y})^2\right] - E[\mathbf{y}^{\rm T}C\mathbf{y}]^2 \\ &= 2\sum_{i=1}^{N} c_{ii}^2\sigma_i^4 + 4\sum_{i,j,i>j} c_{ij}^2\sigma_i^2\sigma_j^2 \\ &= 2\mathbf{\Theta}^{\rm T}D\mathbf{\Theta} \end{aligned} \tag{11}$$

where $d_{ij} = c_{ij}^2$. The variance of φ_1 can be obtained in a similar manner.

$$\begin{aligned} \sigma_{\varphi_1}^2 &= E\left[(x^{\rm T}Cx)^2\right] - E[x^{\rm T}Cx]^2 + 4E\left[(\mathbf{w}^{\rm T}x)^2\right] \\ &= 2\Theta^{\rm T}D\Theta + 4\mathbf{u}^{\rm T}\Theta \end{aligned} \tag{12}$$

where $u_i = w_i^2$. The overall variance of the nonlinear phase noise is then

$$\begin{aligned} \sigma_{\rm NL}^2 &= \gamma^2\left[\sigma_{\varphi_1}^2 + \sigma_{\varphi_2}^2\right] \\ &= \gamma^2[4\Theta^{\rm T}D\Theta + 4\mathbf{u}^{\rm T}\Theta]. \end{aligned} \tag{13}$$

III. DESIGN OF AMPLIFIER GAINS AND SPACINGS FOR PHASE-NOISE MINIMIZATION

With the analytical expressions for $\sigma_{\rm L}^2$ and $\sigma_{\rm NL}^2$, we can study the effect of the system design on the linear and nonlinear phase noises for a fiber link with a fixed length L. We will first study the case when the amplifiers are uniformly spaced and the gain exactly compensates for the signal loss on a per-span basis. The optimal amplifier spacings and hence the optimal N for a fixed L has been studied for various system models addressing various effects, such as the nonlinear parametric gain [18], the post-amplifier loss, the polarization-dependent gain, [19] and others [20], [21]. In this paper, we are mainly concerned with the dependence of the combined-phase-noise variance on the number of amplifiers, since the performance of PSK and DPSK systems is well characterized by the variance of the combined phase noise.

Throughout this paper, unless indicated otherwise, we assume the system parameters shown in Table I.

A. *Uniform Amplifier Spacing With a Per-Span Loss Compensation*

With a per-span compensation and a uniform amplifier spacing along the fiber, it is well known that the noise power and hence the linear phase noise at the receiver is a monotonically decreasing function of the number of amplifiers N [22]. However, the dependence of a nonlinear phase noise on N is different than that of a linear phase noise. Let

$$\sigma_i^2 = \sigma_j^2 = \sigma^2 = b(\mathrm{e}^{\frac{\alpha L}{N}} - 1) \quad (14)$$

$$L_{\mathrm{e},i} = L_{\mathrm{e},j} = L_{\mathrm{e}} = \frac{1-\mathrm{e}^{\frac{-\alpha L}{N}}}{\alpha} \quad (15)$$

$$A_i = A_j = A \quad (16)$$

for all $i, j \in \{1, 2, \ldots, N\}$. In this case

$$\begin{aligned}\phi_{\mathrm{NL}} &= \gamma L_{\mathrm{e}} \sum_{i=1}^{N} \left[\left(A + \sum_{j=1}^{i} x_j \right)^2 + \sum_{j=1}^{i} y_j^2 \right] \\ &= \gamma \left(L_{\mathrm{e}} N A^2 + 2\mathbf{w}^{\mathrm{T}}\mathbf{x} + \mathbf{x}^{\mathrm{T}} C \mathbf{x} + \mathbf{y}^{\mathrm{T}} C \mathbf{y} \right) \quad (17)\end{aligned}$$

where

$$C = L_{\mathrm{e}} \begin{bmatrix} N & N-1 & N-2 & \cdots & 1 \\ N-1 & N-1 & N-2 & \cdots & 1 \\ N-2 & N-2 & N-2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \quad (18)$$

and

$$\mathbf{w} = L_{\mathrm{e}} A [N \; N-1 \; \cdots \; 2 \; 1]. \quad (19)$$

The variance of the nonlinear phase noise is then

$$\begin{aligned}\sigma_{\mathrm{NL}}^2 &= (\gamma L_{\mathrm{e}})^2 \Bigg[4\sigma^4 \left(1^2(2N-1) + 2^2(2N-3) + \cdots + N^2 \right) + 4A^2\sigma^2 \sum_{n=1}^{N} n^2 \Bigg] \\ &= (\gamma L_{\mathrm{e}})^2 \left[4\sigma^4 \sum_{n=1}^{N} n^2 \left(2(N-n) + 1 \right) + 4A^2\sigma^2 \sum_{n=1}^{N} n^2 \right] \\ &= (\gamma L_{\mathrm{e}})^2 \left[4\sigma^4 \left[(2N+1) \sum_{n=1}^{N} (n^2 - 2n^3) \right] + 4A^2\sigma^2 \sum_{n=1}^{N} n^2 \right].\end{aligned}$$

Using the identities

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{n=1}^{N} n^3 = \frac{N^2(N+1)^2}{4}$$

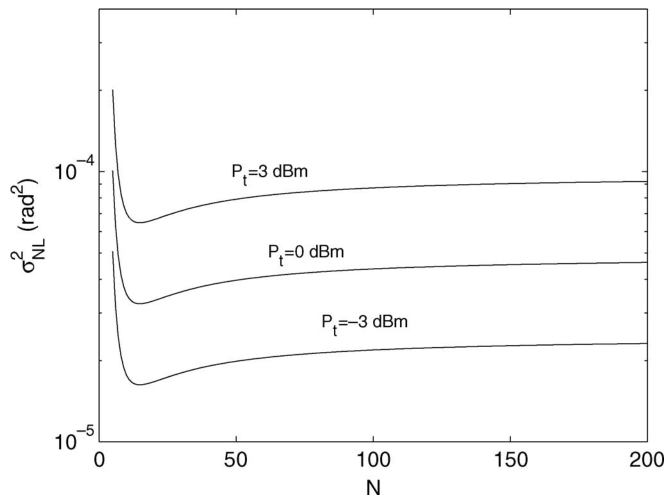

Fig. 2. Dependence of the variance of the nonlinear phase noise on the number of optical amplifiers for a fixed system length $L = 500$ km.

a closed-form expression for the variance of a nonlinear phase noise is given by

$$\begin{aligned}\sigma_{\mathrm{NL}}^2 &= (\gamma L_{\mathrm{e}})^2 \Bigg[4\sigma^4(2N+1)\frac{N(N+1)(2N+1)}{6} - 4\sigma^4(2N+1)\frac{N^2(N+1)^2}{2} + 4A^2\sigma^2 \frac{N(N+1)(2N+1)}{6} \Bigg] \\ &= (\gamma L_{\mathrm{e}})^2 \left[\frac{2N(N+1)(N^2+N+1)}{3}\sigma^4 + \frac{2N(N+1)(2N+1)}{3} A^2 \sigma^2 \right] \\ &= \frac{\gamma^2}{\alpha^2}(1 - \mathrm{e}^{-\frac{\alpha L}{N}})^2 \\ &\quad \times \left[\frac{2N(N+1)(N^2+N+1)}{3} b^2 \left(\mathrm{e}^{\frac{\alpha L}{N}} - 1 \right)^2 + \frac{2N(N+1)(2N+1)}{3} A^2 b \left(\mathrm{e}^{\frac{\alpha L}{N}} - 1 \right) \right]. \quad (20)\end{aligned}$$

The dependence of σ_{NL}^2 on N is shown in Fig. 2 for different values of transmitted power P_t. From the figure, we can see that σ_{NL}^2 is minimized in the vicinity of $N = 15$ for all the power levels and then increases monotonically towards the distributed amplification limit ($N \to \infty$). In addition, as $N \to \infty$, σ_{L}^2 decreases monotonically to $(b\alpha L)/(2A^2)$ [22] since $\sigma^2 \to (b\alpha L)/N$. Therefore, as $L_{\mathrm{e}} \to L/N$, the asymptotic limit of σ_{NL}^2 is

$$\begin{aligned}\lim_{N\to\infty} \sigma_{\mathrm{NL}}^2 &= \gamma^2 \left(\frac{L}{N} \right)^2 \left[\frac{2N^4}{3} \frac{b^2\alpha^2 L^2}{N^2} + \frac{4A^2N^3}{3} \frac{b\alpha L}{N} \right] \\ &= \frac{2\gamma^2 b^2 \alpha^2 L^4}{3} + \frac{4\gamma^2 b\alpha L^3 A^2}{3}. \quad (21)\end{aligned}$$

It should be noted that the overall performance of PSK and DPSK lightwave systems is limited by both the linear phase noise and the nonlinear phase noise. Since they are uncorrelated with each other [11], the variance of the combined noise is simply the sum of the variances, and its dependence on N is shown in Fig. 3. Due to the behavior of the nonlinear phase noise, the variance of the combined noise also exhibits a minimum at $N = 100$. The existence of a minimum combined variance at a finite N has been observed for a wide range of system-parameter values. Therefore, a distributed amplification is actually suboptimal when a nonlinear phase noise is taken into account.

B. Amplifier-Spacing Optimization With a Per-Span Loss Compensation

The mathematical structure of $\phi_{\rm NL}$ can provide further insight into a system design with a per-span compensation. From (17), we see that for $i,\ j \in \{1, 2, \ldots, N\}$ and $i < j$, x_i, y_i appears $j - i$ times more often than x_j and y_j in the total nonlinear phase noise. In other words, the contribution of x_i to $\phi_{\rm NL}$ is greater than x_j for $i < j$. This nonuniformity of the contributions to $\phi_{\rm NL}$ from noises generated at different amplifiers suggests that, for a given N and L, it might be possible to minimize $\sigma^2_{\rm NL}$ by designing σ^2_i to be smaller for smaller values of i. Equivalently, in order to minimize $\sigma^2_{\rm NL}$, the spacing between amplifiers should increase as the signal travels through the fiber. However, it is obvious that, for a linear-phase-noise mitigation, the optimal amplifier spacing with a per-span compensation is uniform; hence, there exists a tradeoff between minimizing linear and nonlinear phase noises. In this case, (14) and (15) become invalid while (16) still holds true. The optimization problem at hand can be formulated as

$$\begin{aligned} \min \quad & \sigma^2_{\rm L} + \sigma^2_{\rm NL} \\ & = 4\gamma^2 \Theta^{\rm T} D \Theta + \left[4\gamma^2 \mathbf{u}^{\rm T} + \frac{1}{2A^2}\mathbf{1}\right]\Theta \\ & = \Theta^{\rm T}\mathbf{H}\Theta + \mathbf{f}\Theta \\ \text{subject to} \quad & \sum_{i=1}^{N} l_i = L, \qquad l_i = \hat{l}_i \\ \text{and} \quad & 0 \le l_i \le L, \qquad i = 1, 2, \ldots, N \end{aligned} \tag{22}$$

where $\mathbf{1} = [1\ 1\ \cdots\ 1]$. For practical transmission links, the spacings l_i and the gains G_i are typically large enough that

$$\sigma^2_i \approx b{\rm e}^{\alpha \hat{l}_i} \quad \text{and} \quad L_{{\rm e},i} \approx \frac{1}{\alpha}. \tag{23}$$

As shown in the Appendix, with this simplification, the objective function becomes a convex function with respect to l_i and $\hat{l}_i$. Therefore, our optimization task is very close to a standard convex optimization problem where efficient methods can be used to find the solution and the global optimality is guaranteed. For a terrestrial fiber link of length $L = 3000$ km with 30 inline amplifiers, the optimal spacing of the amplifiers is shown in Fig. 4. In this example, results of the exact and the convex optimizations [relying on (23)] are indistinguishable. The figure shows that l_i increases with i, in agreement with the intuition developed previously. The reduction in variance is about 11%, and the optimal amplifier spacings range from 85 to 110 km and are not symmetric with respect to the uniform-spacing value of 100 km. For a typical transoceanic system with $L = 10\,000$ km and 200 amplifiers, the optimal amplifier spacing is shown in Fig. 5. The amplifier spacing ranges from 20 km to almost 110 km, and total variance is reduced by 49%. Since the range of values of l_i are not small and the approximation $L_{{\rm e},i} \approx 1/\alpha$ may not be valid, the optimal amplifier spacing for the case without the approximations in (23) is also shown in the figure. It is seen that both problem formulations lead to nearly identical solutions. It should be noted that the effect of a nonlinear phase noise increases with an increasing signal power, whereas the effect of a linear phase noise decreases with an increasing signal power. Therefore, we would expect that the optimal amplifier spacing would be more skewed compared to the uniform spacing when the signal power is high.

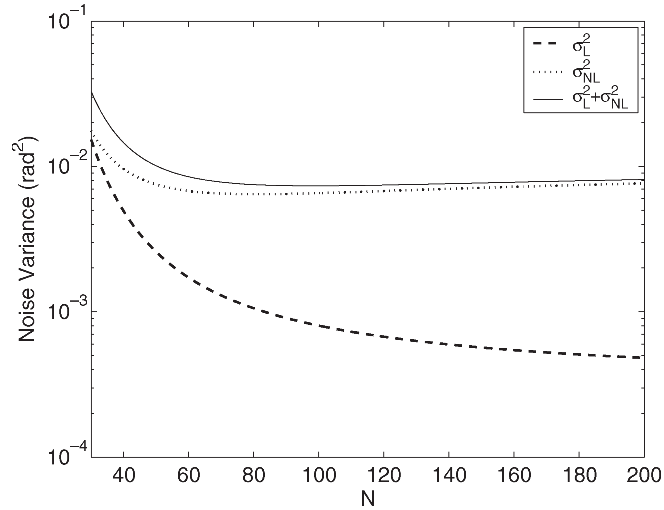

Fig. 3. Variance of the combined linear and nonlinear phase noises as a function of the number of amplifiers N for a fixed system length $L = 3000$ km.

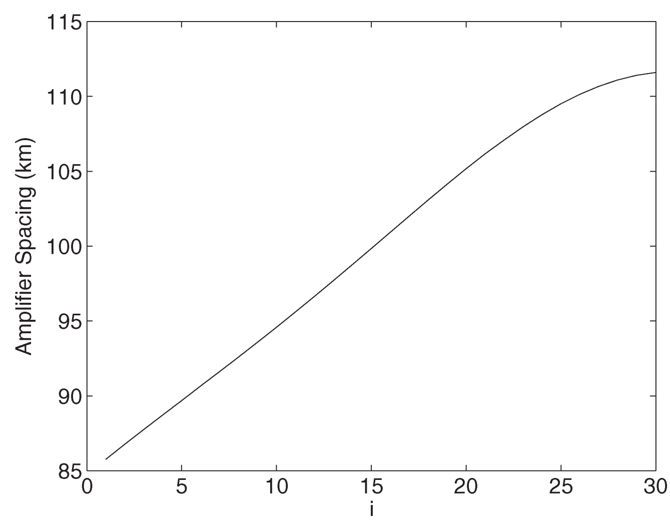

Fig. 4. Optimal amplifier spacings versus amplifier index i in a 3000-km link with 30 amplifiers. Results of the exact and the convex optimizations are indistinguishable. The total-phase-noise variance is reduced by 11%.

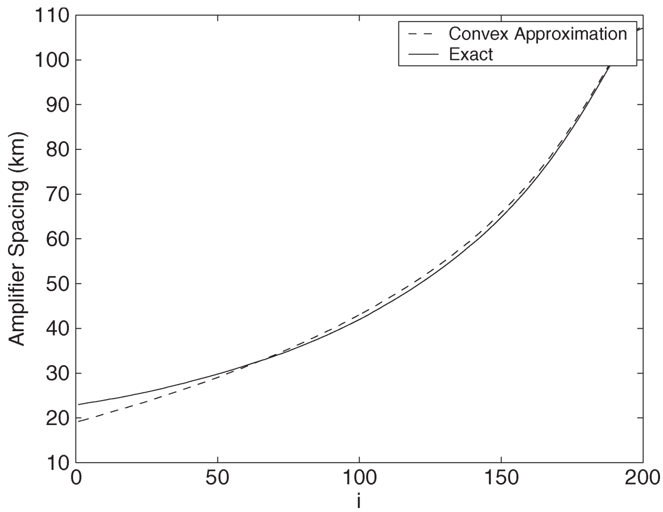

Fig. 5. Optimal amplifier spacings versus amplifier index i in a typical transoceanic fiber link with a total length $L = 10\,000$ km and with $N = 200$ amplifiers. Both exact and convex formulations result in a combined variance reduction of 49%.

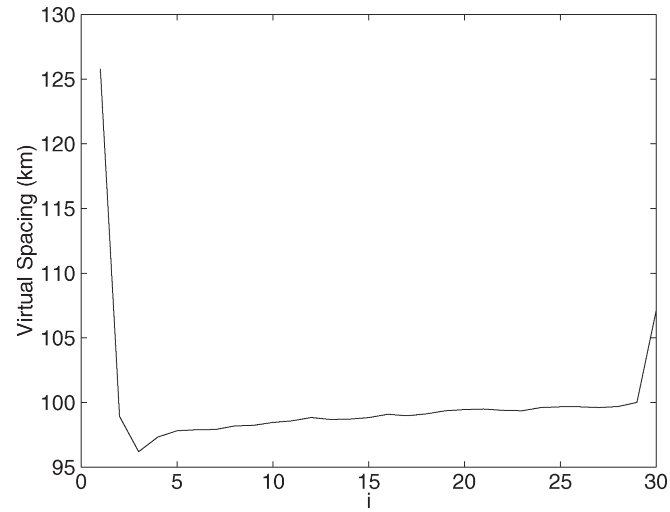

Fig. 6. Optimal virtual spacings between amplifiers versus amplifier index i in a 3000-km link with $N = 30$ amplifiers. The total-phase-noise variance is reduced by 23%.

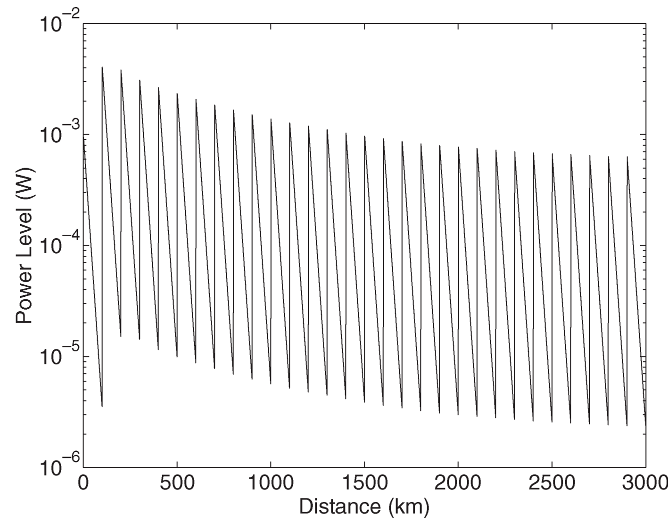

Fig. 7. Signal-power profile under the optimal virtual spacings of Fig. 6 in a 3000-km fiber link with 30 amplifiers.

C. Amplifier Gain Optimization for a Given Amplifier Spacing

We now investigate the performance gain achieved when the spacings of the amplifiers are fixed while constraint of per-span compensation constraint is relaxed. The overall compensation constraint in (1) is maintained. As pointed out in [7], such a scenario is applicable in practice to the phase-noise mitigation in existing terrestrial or transoceanic fiber links where the amplifier spacings are geographically constrained while changing the amplifier gains (hence, virtual spacings) is feasible. In this case, the optimization problem becomes

$$\begin{aligned} \min \quad & \sigma_{\mathrm{L}}^2 + \sigma_{\mathrm{NL}}^2 \\ & = 4\gamma^2 \Theta^{\mathrm{T}} D \Theta + \left[4\gamma^2 \mathbf{u}^{\mathrm{T}} + \frac{1}{2}\mathbf{A} \right] \Theta \\ & = \Theta^{\mathrm{T}} \mathbf{H} \Theta + \mathbf{f} \Theta \\ \text{subject to} \quad & \sum_{i=1}^{N} \hat{l}_i = L \\ \text{and} \quad & 0 \le \hat{l}_i \le L, \qquad i = 1, 2, \ldots, N \qquad (24) \end{aligned}$$

for given amplifier spacings $\{l_1^*, l_2^*, \ldots, l_N^*\}$. To facilitate comparison with other cases, we will assume that the amplifiers are uniformly spaced, i.e., $l_i^* = l_j^* = L/N$. The optimal virtual spacings and hence amplifier gains are shown in Fig. 6. The corresponding signal-power profile as a function of the distance from the transmitter is shown in Fig. 7. The optimal virtual spacing is large in the first amplifier, and it is kept in the vicinity of the per-span compensation value of 100 km until the last amplifier. Such a behavior can be partly explained by the well-known fact that a higher gain early in the fiber reduces the received noise power and, hence, reduces the linear phase noise [23]. We note that the combined variance is reduced by 23%, which is more than the value obtained in the previous section (11%). Therefore, changing the virtual spacing is superior to changing the amplifier spacing both in terms of the practical feasibility and in minimizing the variance of the phase noise. The optimal virtual spacings for a typical transoceanic link is shown in Fig. 8. The shape of the optimal solution resembles that for the terrestrial link. The resulting reduction in phase-noise variance is as much as 81%, which is significantly greater than the value obtained in the previous section (49%).

D. Joint Optimization of Amplifier Spacings and Gains

Last, we will consider the case when we are allowed to vary both the amplifier spacings and the virtual spacings for a phase-noise mitigation.

$$\begin{aligned} \min \quad & \sigma_{\mathrm{L}}^2 + \sigma_{\mathrm{NL}}^2 \\ & = 4\gamma^2 \Theta^{\mathrm{T}} D \Theta + \left[4\gamma^2 \mathbf{u}^{\mathrm{T}} + \frac{1}{2}\mathbf{A} \right] \Theta \\ & = \Theta^{\mathrm{T}} \mathbf{H} \Theta + \mathbf{f} \Theta \\ \text{subject to} \quad & \sum_{i=1}^{N} l_i = L, \qquad \sum_{i=1}^{N} \hat{l}_i = L \\ \text{and} \quad & 0 \le l_i, \hat{l}_i \le L, \qquad i = 1, 2, \ldots, N. \qquad (25) \end{aligned}$$

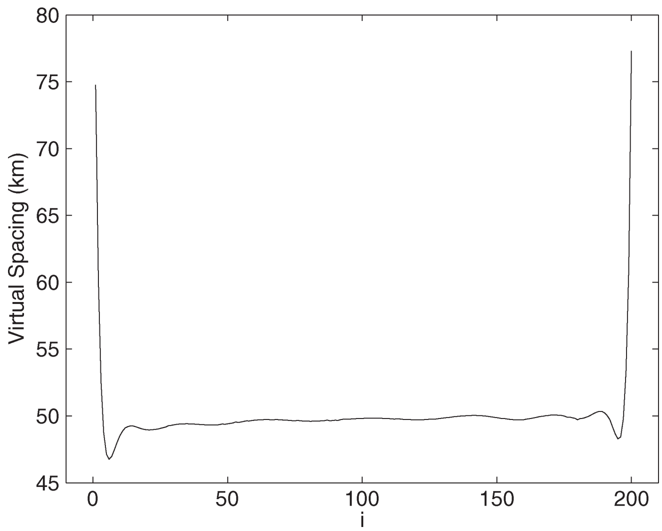

Fig. 8. Optimal virtual spacings between amplifiers versus amplifier index i in a 10 000-km link with $N = 200$ amplifiers. The total-phase-noise variance is reduced by 81%.

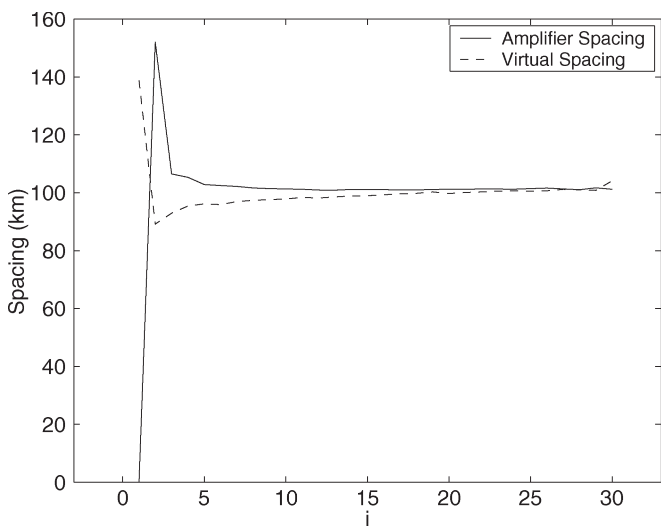

Fig. 9. Optimal amplifier spacings and virtual spacings between amplifiers versus amplifier index i in a 3000-km link with $N = 30$ amplifiers. The total-phase-noise variance is reduced by 45%.

The optimal amplifier spacings and virtual spacings for a link of length 3000 km with 30 amplifiers are shown in Fig. 9. The reduction in the total phase noise is 45%, much greater than the value obtained in the previous two sections (11% and 23%). From the figure, we can see that the first amplifier should have a high gain. This can be explained in terms of minimizing the linear phase noise, as in the last section. The real and virtual spacings converge to 100 km, the uniform amplifier spacing and per-span compensation value, until at the last span where they diverge again. The dependence of the optimal virtual spacing on i is apparently the mirror image of the amplifier spacing. The optimal amplifier spacings and virtual spacings for a transoceanic system are shown in Fig. 10. However, in this case, the variance reduction of 83% is only slightly greater than in the previous section (81%). We note that for both the terrestrial and the transoceanic examples considered, the joint optimization of both amplifier spacings and virtual spacings yields the greatest reduction in $\sigma_{\mathrm{L}}^2 + \sigma_{\mathrm{NL}}^2$ among all the cases considered as one would expect.

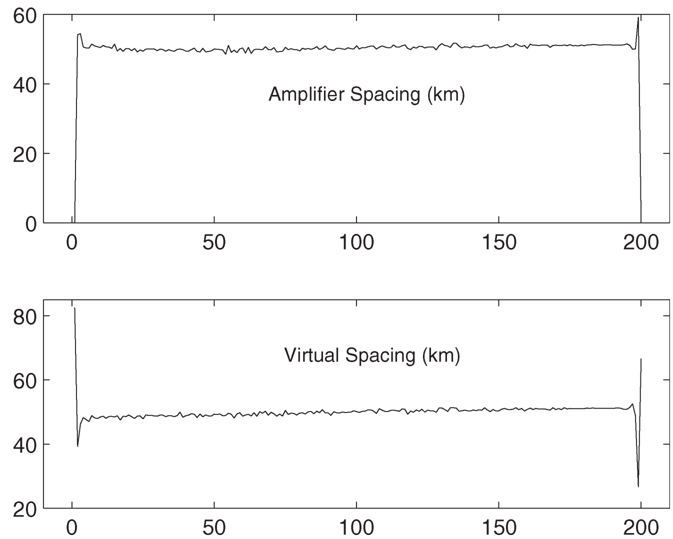

Fig. 10. Optimal amplifier spacings and virtual spacings between amplifiers versus amplifier index i in a 10 000-km link with $N = 200$ amplifiers. The total-phase-noise variance is reduced by 83%.

IV. Conclusion

In this paper, we studied the impact of optical amplifier spacings and gains on linear and nonlinear phase noises in a fiber link of a fixed overall length L. We showed that there exists an optimal finite number of amplifiers N, for which the variance of the nonlinear phase noise is minimized unlike the case of the linear phase noise. We further showed that there exists a finite optimal N that minimizes the variance of the total phase noise (linear plus nonlinear), thus, optimizing performance in systems using PSK or DPSK modulation formats. In addition, for a fixed N, the phase noise can be further minimized when the spacings between the amplifiers are only sum constrained and/or when the per-span loss compensation is not required. The task of obtaining the optimal amplifier spacings and virtual spacings can be approximated as a convex optimization problem, and reductions of 45% and 83% in the combined-phase-noise variance can be achieved in typical terrestrial and transoceanic systems when the optimal amplifier and virtual spacings are used.

Appendix

When

$$\sigma_i^2 \approx b \mathrm{e}^{\alpha \hat{l}_i} \quad \text{and} \quad L_{\mathrm{e},i} \approx \frac{1}{\alpha} \tag{26}$$

the entries of $\mathbf{M}$ is given by

$$M_{ij} = \sqrt{L_{\mathrm{e}}} \frac{A_i}{A_j} = \frac{\prod_{k=j+1}^{i} \mathrm{e}^{\frac{1}{2}\alpha(\hat{l}_k - l_k)}}{\sqrt{\alpha}} = \frac{\mathrm{e}^{\frac{1}{2}\alpha \sum_{k=j+1}^{i}(\hat{l}_k - l_k)}}{\sqrt{\alpha}}$$

and

$$H_{ij} = 4\gamma^2 D_{ij} = 4\gamma^2 C_{ij}^2 = \varepsilon_1 \sum_{k=1}^{N} M_{ki} M_{kj} = \varepsilon_1 \sum_{k=1}^{N} \mathrm{e}^{\beta_k^{\mathrm{T}} \overrightarrow{l}}$$

where $\overrightarrow{l} = [l_1\ l_2\ \cdots\ l_N\ \hat{l}_1\ \hat{l}_2\ \cdots\ \hat{l}_N]^{\mathrm{T}}$ for some constant-valued vector β_k and $\varepsilon_1 > 0$. Therefore, H_{ij} is simply an exponential of a linear combination of the variables. In addition, $\mathbf{w}_i = L_{\mathrm{e}} \sum_{j=i}^{N} (A_j^2/A_i)$ and, hence, $\mathbf{f}_i = \varepsilon_2 \mathbf{w}_i^2 + 1/2\mathbf{A}_i$, $\varepsilon_2 > 0$, are the sums of two exponentials of a linear combination of $\overrightarrow{l}$. Consequently, the objective function $\sigma_{\mathrm{L}}^2 + \sigma_{\mathrm{NL}}^2$ consists of a summation of terms of the forms $H_{ij} b^2 \mathrm{e}^{\alpha \hat{l}_i} \mathrm{e}^{\alpha \hat{l}_j}$ and $f_i b \mathrm{e}^{\alpha \hat{l}_i}$, all of which can be expressed in the form

$$\varepsilon_3 \mathrm{e}^{\beta^{\mathrm{T}} \overrightarrow{l}}$$

with $\varepsilon_3 > 0$. Since the exponential function is convex and the linear transformation and the nonnegative summation preserve convexity, the objective function is convex. With the boundary and sum constraints on l_i, $\hat{l}_i$, the optimization problem is a standard convex optimization problem [24].

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