Fundamental Limit to Delay-Bandwidth Product in One-Dimensional Linear Optical Structures

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Abstract: This delay-bandwidth product is shown to be bounded essentially by the maximum available relative dielectric constant times the length of the structure in wavelengths, and is otherwise independent of structure design and dielectric constant spectrum.

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1. Introduction

Delay of light pulses can be observed in a broad variety of structures and systems, including many based on linear optics, or at least where the response of the system is linear in the signal field (e.g., Refs. [1-3]). A particularly important parameter for applications in optical buffering is the number of pulse widths by which a pulsed signal can be delayed, or equivalently, the delay-bandwidth product of the system. Linear systems for light delay include those based on material dispersion [1,2], and photonic nanostructures, such as linear arrays of coupled resonators [3]. Limits to performance for such linear systems have been considered by several authors [4-6], though generally such limits are derived for specific structures or use concepts such as group velocity that apply to some kinds of structures (e.g., periodic nanostructures and/or materials with linear dispersion) but not always to others (e.g., Fabry-Perot resonators or non-periodic structures [7]). Here we derive an upper bound limit that is completely independent of the form of the device design, or of the shape of the spectra of the materials involved. It applies to one-dimensional structures, such as dielectric stacks or single-mode waveguides, or any system that can be described using an effective one-dimensional wave equation. It covers dielectric constant variations of all kinds, including refractive index, absorption, and/or gain, including even different spectra at different points in the structure.

2. Principle of the limit

The limit we derive is based on a recent theorem [8] that gives a limit to the number of orthogonal functions (or waves) that can be created in a receiving space as a result of linear scattering of an incident field, based only on general bounding properties of the scattering object. The theorem is valid for arbitrarily strong scattering, including multiple scattering, and hence can be applied to high-index contrast dielectric structures or to atomic vapors with very large dielectric constants in specific spectral regions. To see how to apply it to slow light, we need to consider pulse delay in terms of orthogonal functions.

The simplest view is to say that, if a pulse is to be delayed by S time slots, it should appear in its delayed time slot and not in any of the S preceding ones. Hence, we need to be able to control at least $S+1$ independent amplitudes in this output or "receiving" space – one for each time slot to be controlled. Therefore, we need to be able to control the scattering to at least $S+1$ orthogonal functions in the receiving space (see Fig. 1).

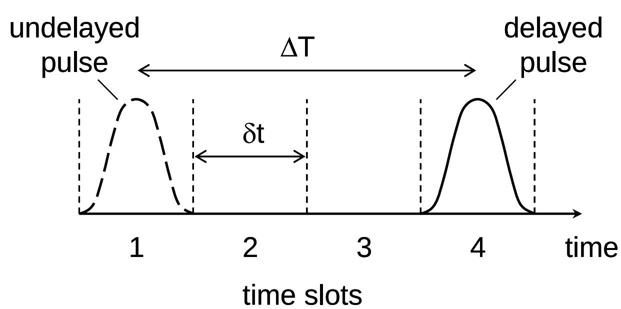

Fig. 1. Illustration of a pulse delayed by 3 time slots so as to appear in the fourth time slot, hence requiring the control of at least 4 independent amplitudes, one for each of the 4 time slots.

The underlying theorem we exploit here [8] states that the number of orthogonal waves that can be generated in a receiving space as a result of linear scattering in a scattering space is

$$M \leq \sqrt{Tr\left(\mathsf{C}^{\dagger}\mathsf{C}\right)Tr\left(\mathsf{G}_S^{\dagger}\mathsf{G}_S\right)} \tag{1}$$

where C is the linear operator that gives effective sources in the scatterer in response to fields in the scatterer (effectively, the dielectric constant in the scatterer), G_S is the Green's function of the free-space wave equation, and the traces (*Tr*) are taken over that space of source functions in the scattering volume that give rise to non-zero waves in the receiving volume. (Formally, M is the number of such waves that are also orthogonal to hypothetical "straight-through" and "single-scattered" waves [8].)

3. Limit for one-dimensional systems

As shown in Ref. [8], we can obtain simple explicit results for the case of one-dimensional systems, i.e., any systems that can be described by a wave equation for a wave of frequency f_o that can be written in the form

$$\frac{d^2\phi}{dz^2} + k_o^2\phi = -k_o^2\eta\left(z, f_o\right)\phi \tag{2}$$

Here $k_o = 2\pi / \lambda_c = 2\pi f_o / v_o$, where v_o is the wave velocity and λ_c is the wavelength, both in the background medium. This is an appropriate equation for electromagnetic waves in one-dimensional problems in isotropic, non-magnetic materials with no free charge or free currents. Then η is the fractional variation in the relative dielectric constant in the structure, i.e.,

$$\eta\left(z, f_o\right) \equiv \frac{\Delta\varepsilon\left(z, f_o\right)}{\varepsilon_{ro}} \tag{3}$$

where ε_{ro} is the background relative dielectric constant, the wave velocity in the background medium is $v_o = c / \sqrt{\varepsilon_{ro}}$, where c is the velocity of light, and for a relative dielectric constant $\varepsilon\left(z, f_o\right)$, we define $\Delta\varepsilon = \varepsilon\left(z, f_o\right) - \varepsilon_{ro}$. (Note that ε may be complex.)

We consider the case where the frequency bandwidth δf of interest to us is much less than the center frequency f_c of the corresponding band. We presume that the slow light structure of interest, of total thickness L much larger than a wavelength in the background medium, will output pulses, either by transmission or reflection, into a receiving space that is correspondingly either "behind" or "in front of" the slow light structure. Then the limit previously derived (see Eqs. (38), (40), (45) and (46) of Ref. [8], considering the case of only one frequency band of interest) says that the maximum number of orthogonal functions into which we can couple, or whose amplitude we can control, in a receiving space of arbitrary length is

$$M_{tot} \leq m_{rt} + \frac{\pi}{\sqrt{3}}\frac{L}{\lambda_c}\eta_{\max} \tag{4}$$

where $\eta_{\max}$ is the largest value of $\left|\eta\left(z, f\right)\right|$ anywhere within the bandwidth of interest and anywhere within the structure, and m_{rt} is 1 for reflection and 2 for transmission. (It is also possible to write somewhat more restrictive limits (e.g., in terms of the average r.m.s. variation in the dielectric constant) if one makes more assumptions about the device structure [8].)

Now, as discussed above, if we wish to say that the pulse is delayed by S time slots, we need to be able to control the amplitudes of at least $S+1$ orthogonal functions. Given that we are interested here in pulses on a carrier frequency f_c, we also have to note that there are two different but almost identical pulses in any given time slot that are formally orthogonal only because they have a carrier phase that differs by 90 degrees. For this reason, we need to double the number of amplitudes we need to control in the empty slots, so that neither of these pulses appears in a give slot. (Likely we do not care about the carrier phase of the pulse in the desired slot, so we need not add in another degree of freedom to control that.) Hence we need to control at least $2S+1$ amplitudes. To achieve this, we must therefore have $M_{tot} \geq 2S+1$. Hence, the maximum time delay $S_{\max}$ in time slots that we can guarantee to have is

$$S_{\max} \leq \frac{\pi}{2\sqrt{3}}\frac{L}{\lambda_c}\eta_{\max} \tag{5}$$

Formally, Eq. (5) is correct for the reflection case, and the limit should be higher by ½ for the transmission case. If we are only interested in cases where we are delaying by multiple pulse widths, for simplicity we can approximately use Eq. (5) in both cases.

Note that this limit does not depend on the bandwidth of signals being considered, provided only that that bandwidth is small compared to the center frequency. We can therefore also derive an approximate limit to the product of delay time and bandwidth. For this, we need to choose a heuristic relation between bandwidth and pulse width since there is no pulse that is simultaneously both bandwidth- and time-limited in the strict sense. We can take the "standard-deviation"-based uncertainty principle limit from Fourier analysis, which gives, for full widths in time (Δt) and in angular frequency ($\Delta\omega$) (i.e., widths that are twice the standard deviation measure of the pulse width in frequency and in time) $\Delta\omega\Delta t \geq 2$, which becomes, for ordinary frequency bandwidth Δf , $\Delta f\,\Delta t \geq 1/\pi$. Then, to be able to delay by a total time of at least ΔT over a frequency bandwidth Δf for even the shortest pulse in that bandwidth, i.e., of duration $\Delta t_{\min} \sim 1/\pi\Delta f$, we need to delay by $S \simeq \Delta T / \Delta t_{\min} = \pi\Delta f\,\Delta T$ time slots. Hence, from Eq. (5),

$$\Delta f\,\Delta T \leq \frac{1}{2\sqrt{3}}\frac{L}{\lambda_c}\eta_{\max} \tag{6}$$

This limit can be shown to be larger than the delay bandwidth corresponding to propagation through a uniform refractive index, and is only slightly larger than the specific limits proposed by Tucker et al. [4] both for ideal dispersive materials and for linear arrays of coupled resonators.

4. Conclusions

The limit, Eq. (6), says that the delay-bandwidth product is essentially bounded by the product of the length of the structure in wavelengths (L/λ_c) times the magnitude of the largest relative dielectric constant variation at any frequency or position in the structure ($\eta_{\max}$). Since this limit grows with dielectric constant rather than refractive index, it leaves open the possibility of particularly large delay-bandwidth products per unit length (i.e., short bit-storage length) in structures with large relative dielectric constants ε_r , such as metals ($\varepsilon_r \sim 100$ in the near infra-red) or atomic vapor systems.

This limit does not show how to design light delay systems, nor does it prove that attaining such a limit is practically possible, especially given loss in such systems. This limit does say that reductions in effective stored pulse lengths below a free-space wavelength require at least proportionate increases in dielectric constants, and the limit is completely independent of the design approach for the one-dimensional structure and the form of the dielectric constant in frequency and position.

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[1] R. M. Camacho, C. J. Broadbent, I. Ali-Khan, and J. C. Howell, "All-Optical Delay of Images using Slow Light," Phys. Rev. Lett. **98**, 043902 (2007)

[2] L. V. Hau, S. E. Harris, Z. Dutton, and C. H. Behroozi, "Light speed reduction to 17 meters per second in an ultracold atomic gas," Nature, **397**, 594–598 (1999).

[3] A. Yariv, Y. Xu, R. K. Lee, and A. Scherer, "Coupled-resonator optical waveguide:a proposal and analysis," Opt. Lett. **24**, 711-713 (1999)

[4] R. S. Tucker, P.-C. Ku, and C. J. Chang-Hasnain, "Slow-Light Optical Buffers: Capabilities and Fundamental Limitations," J. Lightwave Technol. **23**, 4046 – 4066 (2005)

[5] R. W. Boyd, D. J. Gauthier, A. L. Gaeta, and A. E. Willner, "Maximum time delay achievable on propagation through a slow-light medium," Phys. Rev. A**71**, 023801 (2005)

[6] J. B. Khurgin, "Performance limits of delay lines based on optical amplifiers," Opt. Lett. **31**, 948-950 (2006)

[7] M. Gerken and D. A. B. Miller, "Limits to the performance of dispersive thin-film stacks," Applied Optics **44**, No. 18, 3349 – 3357 (2005)

[8] D. A. B. Miller, "Fundamental Limit for Optical Components," J. Opt. Soc. Am. B, published on-line at http://www.opticsinfobase.org/abstract.cfm?msid=76778, January 18, 2007