18th Congress of the International Commission for Optics: Optics for the Next Millennium,, San Francisco, CA (August 3, 1999). Paper [3749-50]. SPIE Vol. 3749, pp110-111 (1999).

Degrees of freedom of an electromagnetic wave

Rafael Piestun and D. A. B. Miller

Ginzton Laboratory, Stanford University, Stanford CA 94305-4085, USA

ABSTRACT

We present a formalism for evaluating the degrees of freedom (d.o.f.) in the communication with electromagnetic waves. We show that, although in principle there are an infinity of d.o.f., the effective number is finite. This is in agreement with the restricted classical theories. We further show that the best transmitting functions are the solutions of a specific eigenvalue equation. The analysis is valid for electromagnetic waves under arbitrary boundary conditions communicating between domains in 3D space.

Keywords: Degrees of freedom, electromagnetic waves, diffraction theory, imaging, microscopy

The evaluation of the information that can be transmitted by an optical system is a problem of primary concern. Classical theories were developed within the scalar and paraxial approximations, considering transmission of information (generally images) between parallel planes of specific optical systems [1,2]. Two main approaches were sought, namely that based on the sampling theory [1], and that based on the theory of the prolate functions [2]. However, many modern photonic systems appear to operate well beyond the limits of validity of such theories. For example, systems utilizing non-paraxial or near-field waves, or involving non-planar or three-dimensional domains cannot be analyzed in this framework. In Ref. [3] the problem was tackled by presenting a scalar theory for evaluating spatial communication channels between volumes in free-space.

In this paper we present a theory for the evaluation of the d.o.f. or communication channels generated by *electromagnetic* wave-fields with *arbitrary* boundary conditions. The conclusions are valid for 1D, 2D, or 3D sources and receivers, as well as for systems working in the near or far-field, and involving more or less complicated architectures. Therefore, the theory can be applied, for example, to imaging systems, optical microscopy, optical interconnects, wave-guided systems, and antennas.

Let us consider a general system, as depicted in Fig. 1, composed of general transmitting sources within a transmitting domain V_T , and a separate receiving domain V_R . In addition, there may be present any material bodies in 3D space. Here, we restrict the analysis to linear homogeneous media and monochromatic radiation.

We consider current [magnetic] sources represented by the complex vector field $\mathbf{J(r')}$ $[\mathbf{M(r')}]$ within V_T . The generated electric and magnetic complex vector fields within V_R are represented as $\mathbf{E(r)}$ and $\mathbf{H(r)}$. The full mathematical description of the problem is given by Maxwell equations. For conciseness we consider only current sources, but the extension to magnetic sources is straightforward. Therefore, we can write a rigorous solution as follows:

$$\mathbf{E}(\mathbf{r}) = \int_{V_T} \Gamma(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')d\mathbf{r}' \tag{1}$$

where $\Gamma(\mathbf{r},\mathbf{r}')$ is the tensor Green's function subject to the appropriate boundary conditions [4].

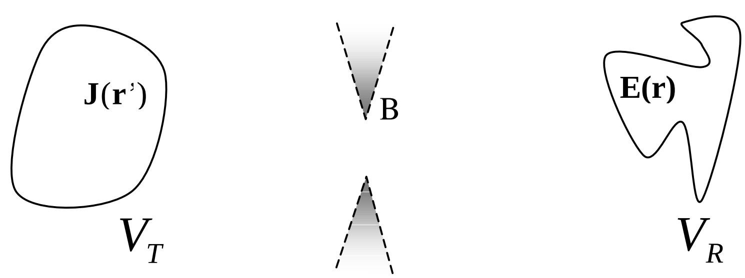

Figure 1: Communication with EM waves between a transmitting and receiving domains in the presence of material bodies (B).

Our first goal is to evaluate the degrees of freedom for communication with EM waves. We consider the space of vectors of functions $[f_1(\mathbf{r})\ \ f_2(\mathbf{r})\ \ f_3(\mathbf{r})]$, where $f_1(\mathbf{r}), f_2(\mathbf{r}), f_3(\mathbf{r}) \in \mathcal{L}_2$; and consider orthonormal bases in the domains V_T and V_R:

$$\mathbf{a}_{T1}(\mathbf{r'}), \mathbf{a}_{T2}(\mathbf{r'}), \cdots \mathbf{a}_{Ti}(\mathbf{r'}), \cdots \qquad \text{and} \qquad \mathbf{a}_{R1}(\mathbf{r}), \mathbf{a}_{R2}(\mathbf{r}), \cdots \mathbf{a}_{Rj}(\mathbf{r}), \cdots \tag{2}$$

respectively. Accordingly we can expand $\mathbf{J(r')}$ and $\mathbf{E(r)}$ as follows

$$\mathbf{J(r')} = \sum_i b_i\, \mathbf{a}_{Ti}(\mathbf{r'}) \qquad\qquad \mathbf{E(r)} = \sum_j d_j\, \mathbf{a}_{Rj}(\mathbf{r}) \tag{3}$$

where b_i and d_j are scalar coefficients. Substitution in Eq. (1) leads to

$$d_j = \sum_i g_{ji} b_i \tag{4}$$

where

$$g_{ji} = \iint_{V_R V_T} \mathbf{a}^{*T}_{Rj}(\mathbf{r})\, \Gamma(\mathbf{r},\mathbf{r'})\, \mathbf{a}_{Ti}(\mathbf{r'})\, d\mathbf{r'} d\mathbf{r} \tag{5}$$

g_{ji} are scalars representing the strength of the coupling connection between wave functions at the transmitting domain and wave functions at the receiving domain. Thus, the amplitude of the receiving function $\mathbf{a}_{Rj}\mathbf{(r)}$ in V_R that results from a source $\mathbf{a}_{Ti}(\mathbf{r'})$ is g_{ji}. For a given system and once the bases are chosen, Eq. (4) gives the basic communications relation between receiving and transmitting waves.

Let us now present a rule for evaluating the connections' strengths. Since $\Gamma(\mathbf{r},\mathbf{r'})$ is continuous except at $\mathbf{r} = \mathbf{r'}$, it can be expanded bilinearly as

$$\Gamma(\mathbf{r},\mathbf{r'}) = \sum_{ij} g_{ji} \mathbf{a}_{Rj}(\mathbf{r}) \otimes \mathbf{a}^{*T}_{Ti}(\mathbf{r'}) \tag{6}$$

where $\otimes$ represents the tensor product. Defining $\|\mathbf{M}\|^2 = \sum_{kl=1}^{3} \left|m^{(kl)}\right|^2$, with $m^{(kl)}$ the elements of $\mathbf{M}$; and integrating over V_T and V_R, leads to

$$\sum_{ij} \left|g_{ji}\right|^2 = \iint_{V_T V_R} \|\Gamma(\mathbf{r},\mathbf{r'})\|^2 d\mathbf{r} d\mathbf{r'}. \tag{7}$$

The integral on the right hand side of Eq. (7) is finite, stating that the total strength of the interconnections is bounded, i.e. the strength of the interconnections is negligible for all but a finite number of d.o.f. Therefore, although in principle the number of d.o.f. is infinite, the number of practically useful channels is finite due to the presence of noise. This result is valid regardless of the specific selected communication functions.

Finally, we find which are the best functions for maximizing the connection strengths between pairs of receiving and transmitting functions. The problem can be reduced to find the *normalized* source $\mathbf{J}(\mathbf{r})$ that maximizes

$$\boldsymbol{\mathcal{E}}_R^2 = (\mathbf{E},\mathbf{E})_{V_R} = \int_{V_R} \mathbf{E}^T(\mathbf{r})\mathbf{E}^*(\mathbf{r}) d\mathbf{r} = \int_{V_T}\int_{V_T} \mathbf{J}^T(\mathbf{r'})\mathbf{K}(\mathbf{r'},\mathbf{r''})\mathbf{J}^*(\mathbf{r''})\, d\mathbf{r'} d\mathbf{r''} \tag{8}$$

where $\mathbf{K}(\mathbf{r'},\mathbf{r''}) = \int_{\mathbf{V_R}} \Gamma^T\mathbf{(r,r')}\Gamma^*(\mathbf{r},\mathbf{r''})d\mathbf{r}$. It can be shown that $\mathbf{K}$ defines a compact, positive, self-adjoint operator. Therefore, it possesses positive eigenvalues μ_n and its eigenvectors form a complete orthogonal set [5], leading to the best transmitting modes. Moreover, if we note $\mu_1 \geq \cdots \geq \mu_n \cdots$, then $\mu_n \xrightarrow{n\to\infty} 0$; and $\mu_1 = \max \boldsymbol{\mathcal{E}}_R$ with normalized sources.

In conclusion, we presented a framework for the study of the d.o.f. of photonic systems, based on a rigorous EM formalism. This theory may prove useful for understanding and designing systems for which previous approaches are not applicable.

Acknowledgements: R. Piestun acknowledges support from the Fulbright Foundation.

REFERENCES

1. D. Gabor, in *Progress in Optics*, E. Wolf, ed. (North-Holland, Amsterdam, 1961), Vol. 1, pp.109-153.
2. G. Toraldo di Francia, J. Opt. Soc. Am. **59**, 799 (1969).
3. D.A.B. Miller, Opt. Lett. **23**, 1645 (1998).
4. R. Harrington, *Time harmonic electromagnetic fields* ( ).
5. D. Porter and D. S. G. Stirling, *Integral equations* (Cambridge U. Press, Cambridge, 1990).